# BIVARIATE DISTRIBUTION

R. SHUNMUGATHAI M.Sc., M.Phil.,

# BIVARIATE DISTRIBUTION

**R. SHUNMUGATHAI** M.Sc., M.Phil., Ph.D.,
Department of Statistics
University of Madras
Chennai – 5

*Masar Printers & Publications*

©Copy right April 2007, Authors.

*Published By*

***Masar Printers & Publications***

*192 Bells Road*

*Chenai -08*

**Price Rs. 250/-**

This book may not be reproduced in any form
without the written permission of the
publisher.

# PREFACE

Life is good for only two things, discovering
Mathematics and teaching Mathematics

The book received a very welcome response from the intended audience – ranging over a wide range of disciplines. The fact that the book was referred to in papers, not only in statistics operations research, engineering and management journals, but also in leading journals in other disciplines, such as, physical Review, J.Math physics, Arch . G. Psychology, Forest science and so on speaks of the wide acceptability, of the book on the one hand, as well as of the growing application of inventory Models in diverse areas, on the other hand.

I have received several favorable communications from interested readers from the world over, from faculty and students to researches, using methods in a wide range of areas. Professor V.S.Sampath kumar have kindly got the book reviewed thoroughly by a learned scholar (whose identity was not known to me); detailed comments and constructive suggestions obtained from the anonymous referee were very useful in carrying out revision of the book. All these have served as valuable feedbacks and have been helpful in my revision work.

Currently in the period of dynamic indeterminism in science, there is hardly a serious piece of research, which, if treated realistically, does not involve operations on stochastic processes. It is thus natural to find a growing awareness and interest in this subject in scientific and technological studies everywhere.

If is felt that, unlike statistical theory, a large variety of introductory and intermediate level texts dedicated to stohasti processes are not available, and that there is need for more books at this level. This book, which aims at the level between that of elementary probability texts and of the excellent advanced works on stochastic processes, the book deals mainly with methods and applications.

It is expected that the book would be suitable as a text book for advanced undergraduate post gradiate and research level curses in statistics , Applied Mathematics, Operations Research, Computer Science, Management Science etc. as also for graduate and beginning research level courses of Study in Pyhsics, Life Sciences and different branches of Engineering. The course could be covered in one or two semesters depending on the interest of the users and on theextent or coverage desired. Researchers who need knowledge of applied probability and applied stochastic processes would also profit from using the book. Some materials, specially on applications from some of the most recent papers are also included. There is somewhat exhaustive chapter on stochastic processes in Queues, and this portion could be used also for a course on queueing theory.

The material presented in the book would, it is hoped, be suitable, in general for students having some mathematical maturity who wish to gain an understanding of the subject and is see how the theory works.

I look forward to receive comments and suggestions on the work from students, teachers and researchers.

I remain intellectually to all those workers whose works have stimulated my interest in the subject. In the present venture, encouragements and suggestions have been received from a number of learned scholars both in this country. I am thankful to them all.

R.SHUNMUGATHAI

Chennai.
April 2007.

# ACKNOWLEDGMENT

I would like to acknowledge several people without them it is impossible even to dream such a work. In particular I am so grateful to all my teachers who taught mathematics, which helped me in discovering such a great subject. I would particularly like to thank Dr. V.S.Sampath Kumar who initiated this work. Also my sincere thanks to Dr. M.R. Srinivasan for their help in various discussions, I had made while preparing this material. The other strength, which makes me to write such a book, is my beloved students who provided me a sympathetic test bed and launching pad when I thought this subject for nearly a decade at different levels and disciplines varying from Management. Also I would like to express my sincere thanks to Dr. V.S. Sampath Kumar for his constant encouragement in my endeavors. Also I use this opportunity to thank the Management of Statistics Department for providing an environment that is so conducive in completing this work successfully. Also my harmonious thanks are due to my friends and my family members for their support and encouragement. I thank my publisher Scitech Publications (India) Private Limited, Chennai for their Cooperation in bringing out this book.

Critical evaluation and suggestions for improvement of the book will be greatefully acknowledged. Please mail our suggestions and feed back shunmuga_77@yahoo.co.in

Chennai.
April 2007.

R.SHUNMUGATHAI

*Dedicated*

*to*

*My Beloved Parents*

# Contents

Page No.

1. INTRODUCTION AND SUMMARY 9-17

2. ABSOLUTELY CONTINUOUS BIVARIATE EXPONENTIAL LOCATION MODEL 18-35

3. ABSOLUTELY CONTINUOUS BIVARIATE EXPONENTIAL SCALE MODEL 36-51

4. ABSOLUTELY CONTINUOUS BIVARIATE EXPONENTIAL LOCATION-SCALE MODEL 52-71

5. RELIABILITY MEASURES OF SYSTEMS WITH ACBVE COMPONENTS 72-86

6. REFERENCES 87-91

# INTRODUCTION AND SUMMARY

## 0.0 Introduction

The univariate exponential distribution has long been the favourite of analysts working with queuing systems, life testing and reliability models. This is because its lack of memory property (LMP) lends itself to easy analysis. Whenever there are two or more variables in a system model, in many analyses they are assumed to be independent. Consider, for example, the inter-arrival and the service times in a queuing system, which are assumed independent in queuing theory while it can be shown that this is not true in most realistic situations. As another example consider the failure analysis of a multi-component system in which the lifetime of one component influences the lifetime of the other components in which case it becomes necessary to introduce a dependence between the variables. To continue to use analytically amenable distributions we will need to assume them to be marginally exponential and introduce multivariate exponential distributions that can model the dependence between the variables. However, unlike the normal distribution, there is no unique natural extension of the univariate exponential distribution and many models have been proposed.

A number of bivariate exponential distributions have been proposed in the literature to describe the failures of two component systems. One such distribution that has received wide acceptance is the bivariate exponential (BVE) of Marshall and Olkin (1967). If X and Y are the lifetimes of two components of a system, then the joint survival function of (X, Y) is,

$$\bar{F}(x, y) = \exp\{-\lambda_1 x - \lambda_2 y - \lambda_{12}(x \vee y)\},\ x,\ y > 0;\ \lambda_1,$$

$\lambda_2 > 0;\ \lambda_{12}$ ? 0.

This distribution is not absolutely continuous but satisfies the bivariate lack of memory property (BLMP),

$$P[X>t_1+s_1, Y>t_2+s_2 \mid X>s_1, Y>s_2] =$$

$$P[X>t_1, Y>t_2],\ t_1, t_2,\ s_1, s_2 > 0.$$

In fact it has been shown that the above is the only bivariate distribution that has exponential marginals and which satisfies BLMP. Properties of this Marshall Olkin model are described in detail in Barlow and Proschan (1975). The BVE enjoys some properties that have useful physical interpretations. It is inappropriate in situations where the two components cannot fail simultaneously with positive probability. For example, in certain types of diseases where the occurrence of simultaneous failure of a pair of organs is rare, an appropriate BVE distribution to describe the failure of these paired organs should be one that is absolutely continuous. Block and Basu (1974) consider a bivariate distribution whose marginals are mixtures of exponentials and having an absolutely continuous joint distribution. Arnold and Strauss (1991) characterize bivariate distributions where the conditional distributions are in prescribed exponential families.

An extension of bivariate exponential distribution has previously been considered by Freund (1961). These discussions have been used in Marshall and Olkin (1967), Block and Basu (1974) and Block (1977). Richard Cowan (1987) proposed a bivariate exponential distribution arising in random geometry. Ryu (1993) proposed an extension of Marshall and Olkin's bivariate exponential distribution. Sarkar (1987) also discussed a continuous bivariate exponential distribution. Absolutely continuous bivariate exponential (ACBVE) models have been widely used in the analysis of competing risks data involving two risk components. For such an analysis, frequentist approach often runs into difficulty due to a likelihood containing some nonidentifiable parameters. With an aim to overcome this nonidentifiability Wang and Ghosh (2000) proposed a theory based on Bayesian procedures.

In this dissertation we are considering location, scale and location-scale families arising out of ACBVE distribution proposed by Block and Basu (1974) with equal marginals. We derive MREE, UMVUE and MLE of location, scale and location-scale parameters. Further optimal tests and reliability performance measures are obtained. Lehmann and Casella (1998) discussed the problem of equivariant estimation of location and scale parameters. Edwin Prabakaran and Chandrasekar (1994) proposed simultaneous equivariant estimation of the location–scale parameter in the location-scale models. Lehmann and Casella (1998) discussed the problem of simultaneous optimal unbiased estimator. Kale and Chandrasekar (1983) and Chandrasekar (1984) show that M-optimality, D-optimality, T-optimality and $Q_A$-**optimality criteria lead to the same optimal vector unbiased estimator.**

We derive optimal tests for hypotheses concerning location and scale parameters. A likelihood ratio test for absolutely continuous bivariate exponential model has been proposed by Gupta, Mehrotra and Michalek (1984). Hanagal and Kale (1991) have discussed large sample tests for an absolutely continuous bivariate exponential model. Hanagal (1992) discussed some inference results in modified Freund's bivariate exponential distribution. Some inference results in an absolutely continuous multivariate exponential model of Block have been discussed by Hanagal (1993). Usha and Subramanyam (2005) proposed tests of independence in a bivariate exponential distribution.

In the last chapter we discuss the reliability performance measures and their estimation for two-unit systems with ACBVE joint distribution. Hanagal (1995) discussed the problem of estimation of reliability in absolutely continuous bivariate exponential stress-strength models. Estimation of system reliability from stress-strength relationship is discussed by Hanagal (1996). Hanagal (1998) discussed the estimation of system reliability in stress-strength models for distributions useful in life testing. Estimation of reliability of a component subjected to bivariate exponential stress has been discussed by Hanagal (1999). Recently Chandrasekar and Amala Revathy (2007) derived equivariant estimation of parameters based on sequential order statistics for (1, 3) and (2, 3) systems.

## 0.1 Summary

This dissertation deals with the location, scale and location-scale families arising out of ACBVE distribution with equal marginals. The joint pdf of ACBVE with equal marginals is

$$f(x, y) = \frac{(\alpha+\beta)(2\alpha+\beta)}{2} \exp\{-\alpha(x+y) - \beta(x \vee y)\} \ ,$$

$$x, y > 0, \ \ \alpha, \beta > 0.$$

The location-scale ACBVE has the pdf

$$fx_{,}\,t(x, y) =$$

$$\frac{(\alpha+\beta)(2\alpha+\beta)}{2\tau^2} \exp\left[-\frac{1}{\tau}\{\alpha(x+y) + \beta(x \vee y) - (2\alpha+\beta)\xi\}\right]$$

$$x, y > \xi, \ \xi \in R, \ \tau > 0$$

Throughout our dissertation, it is assumed that $\alpha$, $\beta$ are known. When $\tau = 1$, the resulting family is the ACBVE location family. When $\xi = 0$, the resulting family is the ACBVE scale family.

Equivariant estimation of location, scale, and location-scale parameters are discussed. We derive the optimal tests concerning location and scale parameters. Some discussions about reliability performance measures and their estimation also have been included.

The dissertation consists of four chapters in addition to the present chapter on Introduction and Summary. In each chapter the first section provides a summary of the corresponding chapter.

Chapter 1 deals with the location family arising out of ACBVE distribution with equal marginals. This chapter includes some results similar to those available in Block and Basu (1974), Lehmann (1986) and Lehmann and Casella (1998). Section 1.1 provides some definitions and notations required in this chapter. Some relevant distributional results on ACBVE are derived in Section 1.2. In Section 1.3 we discuss the estimation of location parameter. Section 1.4 deals with the equivariant estimation of the location parameter of an ACBVE belonging to a location family when the loss is i) squared error and ii) absolute error. The Pitman estimator is also derived. Some particular cases are discussed in this section. In Section 1.5 we derive optimal tests for hypotheses concerning the location parameter.

Chapter 2 deals with the scale family arising out of ACBVE distribution with equal marginals. The results of this chapter are based on Block and Basu (1974), Lehmann (1986) and Lehmann and Casella (1998). Section 2.1 provides some definitions and notations required in this chapter. Some relevant distributional results on ACBVE are discussed in Section 2.2. In Section 2.3 we discuss the UMVUE estimation of the scale parameter. Section 2.4 deals with the equivariant estimation of the scale parameter of an ACBVE belonging to a scale family. The Pitman estimator is derived. Some particular cases are also discussed in this section. In Section 2.5 we derive optimal tests for hypothesis concerning the scale parameter.

Chapter 3 deals with the location-scale family arising out of ACBVE distribution with equal marginals. This chapter includes some results similar to those available in Block and Basu (1974), Sukhatme (1937), Kale and Chandrasekar (1983), Chandrasekar (1984), Edwin Prabakaran and Chandrasekar (1994), Amala Revathy (2003) and Lehmann and Casella (1998). Section 3.1 provides some definitions and notations required in this chapter. Some relevant distributional results on ACBVE are discussed in Section 3.2. In Section 3.3 we discuss the estimation of UMRUE of the location-scale parameter. Section 3.4 deals with the simultaneous equivariant estimation of the location–scale parameter of an ACBVE model belonging to a location-scale family. Some particular cases are also discussed. In Section 3.5 we discuss optimal estimation of the population percentiles.

Chapter 4 deals with the reliability performance measures for two-unit systems when the failure times follow ACBVE distribution with equal marginals. This chapter includes results similar to those available in Block and Basu (1974), Edwin Prabakaran and Chandrasekar (1994), Chandrasekar and Amala Revathy (2007) and Lehmann and Casella (1998). Section 4.1 provides some definitions and notations required in this chapter. Some distributional results are discussed in Section 4.2. In Section 4.3 we obtain the reliability performance measures of two component parallel, series and standby systems when the failure times follow ACBVE distribution with equal marginals. In Section 4.4 we discuss optimal estimation of reliability performance measures for two component parallel, series and standby systems when the failure times follow ACBVE distribution with equal marginals.

# ABSOLUTELY CONTINUOUS BIVARIATE EXPONENTIAL LOCATION MODEL

## 1.0 Summary

Block and Basu (1974) proposed absolutely continuous bivariate exponential (ACBVE) distribution and derived some distributional results. In this chapter we consider a location family arising out of ACBVE with equal marginals. Lehmann and Casella (1998) provide a detailed discussion on location family and equivariant estimation of location parameter. In this chapter we discuss the problem of equivariant estimation of the location parameter of ACBVE model belonging to a location family.

Section 1.1 provides some definitions and notations required in this chapter. Some distributional results on ACBVE are discussed in Section 1.2. In Section 1.3 Uniformly Minimum Variance Unbiased Estimator (UMVUE) of the location parameter is obtained. Section 1.4 deals with the equivariant estimation of the location parameter of an ACBVE belonging to a location family when the loss is squared

error and absolute error. The Pitman estimator is also obtained. Some particular cases are discussed. In Section 1.5 we derive optimal tests for hypotheses concerning the location parameter.

## 1.1 Preliminaries

Let X be a random variable (vector) with the distribution function

$F_{\xi}(.)$, $\xi \in R$. Let $\{F_{\xi}, \xi \in R\}$

be a location family so that $F_{i}(x) = F(x - \hat{\imath})$ for some distribution function F.

In this chapter we are interested in the equivariant estimation of the location parameter ξ. Following Lehmann and Casella (1998) minimum risk equivariant estimator (MREE) of ξ is given by

$$\delta^{*}(x) = \delta_{0}(x) - \upsilon^{*}(y),$$

where $\delta_0$ is a location equivariant estimator, $\upsilon^{*}$ minimizes

$$E_{0}\{\rho(\delta_{0} - \upsilon(y)) \,|\, Y = y\}$$

with respect to u and the loss function is

$$L(\xi; \delta) = \rho(\delta - \xi), \text{ for some } \rho.$$

Here $\mathbf{y} = (y_1, y_2, \ldots, y_{n-1})$, where $y_i = x_i - x_n$, $i = 1, 2, 3, \ldots, n-1$.

**Definition 1.1.1**

A random vector (X, Y) is said to follow ACBVE with equal marginals if its pdf is of the form,

$$f(x,y)=\frac{(\alpha+\beta)(2\alpha+\beta)}{2}\exp\{-\alpha(x+y)-\beta(x\vee y)\}\ ,\ x,\ y>0,$$

$\alpha, \beta > 0$.

For fixed $(\alpha, \beta)$, the following pdf provides a location family.

Let

$$f_\xi(x,y)=\frac{(\alpha+\beta)(2\alpha+\beta)}{2}\exp[-\{\alpha(x+y)+\beta(x\vee y)-(2\alpha+\beta)\xi\}],$$

$x, y > \xi;\ \xi \in R.$ ...(1.1.1)

For fixed $(\alpha, \beta)$, the distribution of $(X - \xi, Y - \xi)$ does not depend on $\xi$. Therefore, when $(\alpha, \beta)$ is known; the family in (1.1.1) is a location family with the location parameter $\xi$. The transformation $(X^*, Y^*) = (X + a, Y + a)$ induces the transformation $\xi^* = \xi + a$ on the parameter space and the model is invariant. Throughout this chapter it is assumed that $(\alpha, \beta)$ is known.

## 1.2 Distributional Results

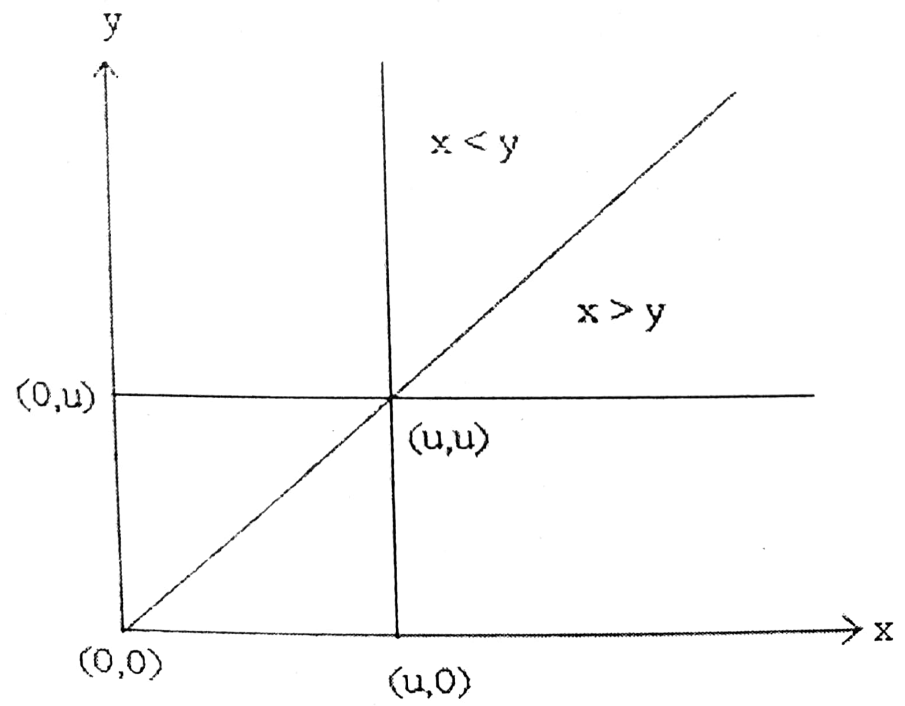
**Theorem 1.2.1** Let (X, Y) have the pdf given in (1.1.1). Then

$$(X \wedge Y) \sim E\left(\xi, \frac{1}{(2\alpha+\beta)}\right)$$

Here E (a, b) refers to a random variable with pdf $\left(\frac{1}{b}\right)\exp\left\{-\frac{1}{b}(x-a)\right\}$, $x > a;\ a \in R,\ b > 0$.

**Proof** For $u > \xi$ $P_\xi[X \wedge Y > u] = P_\xi[X > u, Y > u]$

$$\frac{(\alpha+\beta)(2\alpha+\beta)}{2}\int_u^\infty\int_u^\infty \exp\left[-\{\alpha(x+y)+\beta(x \vee y)-(2\alpha+\beta)\xi\}\right]dxdy$$



$$= \frac{(\alpha+\beta)(2\alpha+\beta)}{2} \int_u^\infty \left\{ \int_u^x \exp[-\{\alpha(x+y)+\beta x-(2\alpha+\beta)\xi\}]dy \right\} dx +$$

$$\frac{(\alpha+\beta)(2\alpha+\beta)}{2} \int_u^\infty \left\{ \int_x^\infty \exp[-\{\alpha(x+y)+\beta y-(2\alpha+\beta)\xi\}]dy \right\} dx$$

$$= \frac{(\alpha+\beta)(2\alpha+\beta)}{2} \left[ \frac{1}{\alpha} \int_u^\infty \exp\{-(\alpha+\beta)x-\alpha u+(2\alpha+\beta)\xi\}dx \right.$$

$$-\frac{1}{\alpha}\int_u^\infty \exp\{-(2\alpha+\beta)(x-\xi)\}dx +$$

$$\frac{(\alpha+\beta)(2\alpha+\beta)}{2} \left[ \frac{1}{\alpha(\alpha+\beta)} \exp\{-(2\alpha+\beta)(u-\xi)\} \right.$$

$$-\frac{1}{\alpha(2\alpha+\beta)} \exp\{-(2\alpha+\beta)(u-\xi)\}$$

$$\left. + \frac{1}{(\alpha+\beta)(2\alpha+\beta)} \exp\{-(2\alpha+\beta)(u-\xi)\} \right]$$

$$= \frac{(\alpha+\beta)(2\alpha+\beta)}{2} [\exp\{-(2\alpha+\beta)(u-\xi)\}] \left\{ \frac{(2\alpha+\beta)-(\alpha+\beta)+\alpha}{\alpha(\alpha+\beta)(2\alpha+\beta)} \right\}$$

$$= \exp\{-(2\alpha+\beta)(u-\xi)\}.$$

Therefore, $P_\xi[X \wedge Y > u] = \exp\{-(2\alpha+\beta)(u-\xi)\}$, $u > \xi$.

Hence $(X \wedge Y) \sim E\left(\xi, \frac{1}{(2\alpha+\beta)}\right)$.

### Remark 1.2.1

$(2\alpha+\beta)(X \wedge Y - \xi)$ follows standard exponential distribution.

### Theorem 1.2.2

Let $(X_i, Y_i)$, $i = 1, 2, 3, \ldots, n$ be a random sample from a distribution with the pdf given in (1.1.1). Then $T = \underset{1 \le i \le n}{\mathrm{Min}}(X_i \wedge Y_i)$ is a complete sufficient statistic.

**Proof** The joint pdf of the sample is

$p_\xi(x,y)$

$$\left\{\frac{(\alpha+\beta)(2\alpha+\beta)}{2}\right\}^n \exp\left[-\sum_i \{\alpha(x_i+y_i)+\beta(x_i \vee y_i)-(2\alpha+\beta)\xi\}\right],$$

$x_i \wedge y_i > \xi$, for every i.

$$= \left\{\frac{(\alpha+\beta)(2\alpha+\beta)}{2}\right\}^n \exp\left[-\sum_i \{\alpha(x_i+y_i)+\beta(x_i \vee y_i)-(2\alpha+\beta)\xi\}\right],$$

$\underset{i}{\mathrm{Min}}(x_i \wedge y_i) > \xi.$

Therefore by Factorization theorem,

$T = \underset{i}{\mathrm{Min}}(X_i \wedge Y_i)$ is sufficient.

By Theorem 1.2.1, $(X_i \wedge Y_i)$, i= 1, 2, 3, …, n are i.i.d with $E\left(\xi, \frac{1}{(2\alpha+\beta)}\right)$ common distribution. Thus T ~ $E\left(\xi, \frac{1}{n(2\alpha+\beta)}\right)$. Hence T is complete.

## MLR Property

The joint pdf of the sample from (1.1.1) is

$p_\xi(\mathbf{x},\mathbf{y})=$

$$\left\{\frac{(\alpha+\beta)(2\alpha+\beta)}{2}\right\}^n \begin{cases} \exp\left[-\sum_i \{\alpha(x_i+y_i)+\beta(x_i \vee y_i)-(2\alpha+\beta)\xi\}\right], \\ \underset{i}{\mathrm{Min}}\,(x_i \wedge y_i) > \xi. \\ \qquad 0, \text{ otherwise.} \end{cases}$$

... (1.2.1).

Take $\xi_1$, $\xi_2$ such that $\xi < \xi_1 < \xi_2$. Then the support of $\{P_\xi, \xi = \xi_1, \xi_2\}$ is

$$(\xi_1, ?) \cup (\xi_2, ?) = (\xi_1, ?).$$

$$\frac{p_{\xi_2}(\mathbf{x},\mathbf{y})}{p_{\xi_1}(\mathbf{x},\mathbf{y})} =$$

$$\begin{cases} 0, \quad \text{if } \xi_1 < \underset{i}{\mathrm{Min}}\,(x_i \wedge y_i) < \xi_2. \\ \exp\{n(2\alpha+\beta)(\xi_2-\xi_1)\}, \quad \text{if } \xi_2 < \underset{i}{\mathrm{Min}}\,(x_i \wedge y_i) < \infty. \end{cases}$$

Therefore the family of distributions of **(X, Y)** possesses MLR property in $\underset{i}{\mathrm{Min}}\,(X_i \wedge Y_i)$.

### 1.3 UMVUE

We wish to find the UMVUE of the location parameter $\xi$. Consider $T = \underset{1\le i\le n}{\mathrm{Min}}(X_i \wedge Y_i)$. From Theorem 1.2.2, T is complete sufficient.

Also $T \sim E\left(\xi, \frac{1}{n(2\alpha+\beta)}\right)$.

Thus $E_\xi(T) = \xi + \frac{1}{n(2\alpha+\beta)}$.

Therefore

$$E_\xi\left(T - \frac{1}{n(2\alpha+\beta)}\right) = \xi \quad \text{for every } \xi.$$

Since T is complete sufficient $\delta^* = \left(T - \frac{1}{n(2\alpha+\beta)}\right)$ is the UMVUE of $\xi$.

**Maximum Likelihood Estimator**

It is easy to observe that the MLE of $\xi$ is $T = \underset{i}{\mathrm{Min}}(X_i \wedge Y_i)$. Clearly MLE is not unbiased. Further it is consistent for $\xi$, but not a CAN estimator.

## 1.4 Equivariant Estimation

Let $(X_i, Y_i)$, $i = 1, 2, 3, \ldots, n$ be a random sample from a distribution with the pdf given in (1.1.1) so that the joint pdf is given by (1.2.1). Clearly the distribution of $(\mathbf{X}, \mathbf{Y})$ belongs to a location family with $\xi$ as the location parameter. Assuming that $(\alpha, \beta)$ is known, we wish to find the MREE of $\xi$.

Take $\delta_0(\mathbf{x}, \mathbf{y}) = \operatorname*{Min}_i (x_i \wedge y_i)$.

Consider

$$\begin{aligned}\delta_0(a + \mathbf{x}, a + \mathbf{y}) &= \operatorname*{Min}_i \{(x_i + a) \wedge (y_i + a)\} \\ &= \operatorname*{Min}_i \{(x_i \wedge y_i) + a\} \\ &= \operatorname*{Min}_i (x_i \wedge y_i) + a \\ &= \delta_0(\mathbf{x}, \mathbf{y}) + a.\end{aligned}$$

This implies that $\operatorname*{Min}_i (x_i \wedge y_i)$ is a location equivariant estimator. From Theorem 1.2.2,

$T = \operatorname*{Min}_i (X_i \wedge Y_i)$ is a complete sufficient statistic.

Following Lehmann and Casella (1998), the MREE with respect to the squared error loss is

$$\delta^*(\mathbf{x}, \mathbf{y}) = \delta_0(\mathbf{x}, \mathbf{y}) - E_0(\delta_0).$$

When $\xi = 0$,

$$\delta_0(\mathbf{X},\mathbf{Y}) = \underset{i}{\text{Min}}\,(X_i \wedge Y_i) \sim E\left(0, \frac{1}{n(2\alpha+\beta)}\right).$$

Then, for squared error loss function,

$$E_0\,\{\delta_0\,(\mathbf{X},\,\mathbf{Y})\} = \frac{1}{n(2\alpha+\beta)}.$$

Thus, the MREE of $\xi$ is given by

$$\delta^*\,(\mathbf{X},\,\mathbf{Y}) = \underset{i}{\text{Min}}\,(X_i \wedge Y_i) \quad \frac{1}{n(2\alpha+\beta)}. \qquad \ldots\,(1.4.1)$$

**Pitman form**

The Pitman form for MREE of $\xi$ with respect to squared error loss is

$$\delta^*\,(\mathbf{x},\,\mathbf{y}) = \frac{\int_{-\infty}^{\delta_0} u\,f\,(x-u,\,y-u)\,du}{\int_{-\infty}^{\delta_0} f\,(x-u,\,y-u)\,du}$$

$$= \frac{\int_{-\infty}^{\delta_0} u\left\{\frac{(\alpha+\beta)(2\alpha+\beta)}{2}\right\}^n \exp\left[-\sum_i \{\alpha(x_i+y_i)+\beta(x_i \vee y_i)-(2\alpha+\beta)u\}\right]du}{\int_{-\infty}^{\delta_0} \left\{\frac{(\alpha+\beta)(2\alpha+\beta)}{2}\right\}^n \exp\left[-\sum_i \{\alpha(x_i+y_i)+\beta(x_i \vee y_i)-(2\alpha+\beta)u\}\right]du}$$

$$= \frac{\int_{-\infty}^{\delta_0} u\,\exp\{n\,(2\alpha+\beta)\,u\}\,du}{\int_{-\infty}^{\delta_0} \exp\{n\,(2\alpha+\beta)\,u\}\,du}$$

$$= \frac{\delta_0 \exp\{n(2\alpha+\beta)\delta_0\}}{\exp\{n(2\alpha+\beta)\delta_0\}} - \frac{\exp\{n(2\alpha+\beta)\delta_0\}}{n(2\alpha+\beta)\exp\{n(2\alpha+\beta)\delta_0\}}$$

$$= \delta_0 - \frac{1}{n(2\alpha+\beta)}.$$

i.e., $$\delta^*(\mathbf{X}, \mathbf{Y}) = \operatorname*{Min}_i (X_i \wedge Y_i) - \frac{1}{n(2\alpha+\beta)}.$$

This coincides with (1.4.1).

**Particular cases**

(i) Suppose that $\alpha = 1, \beta = 1$.

Then MREE of $\xi$ is given by

$$\delta^*(\mathbf{X}, \mathbf{Y}) = \operatorname*{Min}_i (X_i \wedge Y_i) - \frac{1}{3n}.$$

(ii) Suppose that $\alpha = 1, \beta = 2$.

Then MREE of $\xi$ is given by

$$\delta^*(\mathbf{X}, \mathbf{Y}) = \operatorname*{Min}_i (X_i \wedge Y_i) - \frac{1}{4n}.$$

(iii) Suppose that $\alpha = 2, \beta = 1$.

Then MREE of $\xi$ is given by

$$\delta^*(\mathbf{X}, \mathbf{Y}) = \operatorname*{Min}_i (X_i \wedge Y_i) - \frac{1}{5n}.$$

**MREE with respect to absolute error loss**

Let us now find MREE with respect to absolute error loss. Following Lehmann and Casella (1998)

$$\delta^{*}(x) = \delta_0 - \mathrm{Med}_0(\delta_0).$$

**When** $\xi = 0$,

$$\delta_0 \sim E\left(0, \frac{1}{n(2\alpha+\beta)}\right).$$

Median Q is given by

$$n(2\alpha+\beta)\int_0^Q \exp\{-n(2\alpha+\beta)x\}dx = \frac{1}{2}$$

$$\Rightarrow 1 - \exp\{-n(2\alpha+\beta)Q\} = \frac{1}{2}$$

$$\Rightarrow \exp\{-n(2\alpha+\beta)Q\} = \frac{1}{2}$$

$$\Rightarrow n(2\alpha+\beta)Q = \log 2.$$

Therefore, $Q = \dfrac{1}{n(2\alpha+\beta)} \log 2.$

Thus, MREE of $\xi$ with respect to absolute error loss is given by

$$\delta^*(\mathbf{X}, \mathbf{Y}) = \operatorname*{Min}_i (X_i \wedge Y_i) - \frac{1}{n(2\alpha+\beta)} \log 2.$$

**Particular cases**

(i) Suppose that $\alpha = 1, \beta = 1$.

Then MREE of $\xi$ is given by

$$\delta^*(\mathbf{X}, \mathbf{Y}) = \operatorname*{Min}_i (X_i \wedge Y_i) - \frac{1}{3n} \log 2.$$

(ii) Suppose that $\alpha = 1, \beta = 2$.

Then MREE of $\xi$ is given by

$$\delta^*(\mathbf{X}, \mathbf{Y}) = \operatorname*{Min}_i (X_i \wedge Y_i) - \frac{1}{4n} \log 2.$$

(iii) Suppose that $\alpha = 2, \beta = 1$.

Then MREE of $\xi$ is given by

$$\delta^*(\mathbf{X}, \mathbf{Y}) = \operatorname*{Min}_i (X_i \wedge Y_i) - \frac{1}{5n} \log 2.$$

## 1.5 UMP Tests

### One sided test

We are interested in testing

$$H : \xi \le \xi_0 \text{ vs } K : \xi > \xi_0 .$$

From Section 1.2, we have seen that the family $\{p_\xi(\mathbf{X}, \mathbf{Y}), \xi \in R\}$ possesses MLR property in $T(\mathbf{X}, \mathbf{Y}) = \operatorname*{Min}_i (X_i \wedge Y_i)$.

Thus, for testing H: $\xi \leq \xi_0$ vs K: $\xi > \xi_0$ there exists a UMP level α test given by

$$\Phi^*(x, y) = \begin{cases} 1, & t > c. \\ 0, & t < c. \end{cases}$$

where c is determined by

$$E_{\xi_0}\left\{\Phi^*(X, Y)\right\} = \alpha.$$

Thus c is given by

$$n(2\alpha+\beta)\int_c^{\infty} \exp\{-n(2\alpha+\beta)(x-\xi_0)\}dx = \alpha.$$

Thus $c = -\dfrac{\log\alpha}{n(2\alpha+\beta)} + \xi_0$.

**Two sided test**

We are interested in testing

$$H: \xi = \xi_0 \text{ vs } K: \xi \neq \xi_0.$$

From Section 1.2, we have seen that the family $\{f_\xi(\mathbf{X}, \mathbf{Y}), \xi \in R\}$ possesses MLR

property in $T(\mathbf{X}, \mathbf{Y}) = \underset{i}{\text{Min}}(X_i \wedge Y_i)$.

Thus, for testing $H: \xi = \xi_0$ vs $K: \xi < \xi_0$ there exists a

UMP level α test is given by

$$\Phi_1^*(x, y) = \begin{cases} 1, & t < c_1. \\ 0, & \text{otherwise.} \end{cases}$$

where $c_1$ is such that $P_{\xi_0}(T < c_1) = \alpha$.

Recall that, $n(2\alpha+\beta)(T - \xi) \sim E(0, 1)$.

Therefore, $c_1$ is such that

$$P_{\xi_0}\{n(2\alpha+\beta)(T-\xi_0) < n(2\alpha+\beta)(c_1-\xi_0)\} = \alpha$$

$$\text{or} \quad P_{\xi_0}\{n(2\alpha+\beta)(T-\xi_0) > n(2\alpha+\beta)(c_1-\xi_0)\} =$$

$1-\alpha$.

$$\text{i.e., } \exp\{-n(2\alpha+\beta)(c_1-\xi_0)\} = 1-\alpha.$$

$$\text{Thus, } c_1 = \xi_0 - \frac{\log(1-\alpha)}{n(2\alpha+\beta)}.$$

For testing $H: \xi = \xi_0$ vs $K: \xi > \xi_0$ there exists a UMP level $\alpha$ test and is given by

$$\Phi_2^*(\mathbf{x}, \mathbf{y}) = \begin{cases} 1, & t > c_2 . \\ 0, & \text{otherwise.} \end{cases}$$

where $c_2$ is such that $P_{\xi_0}(T > c_2) = \alpha$.

Therefore, $c_2 = \xi_0 - \dfrac{\log \alpha}{n(2\alpha+\beta)}$.

Define $\Phi^*(\mathbf{x}, \mathbf{y}) = \begin{cases} 1, & t > \xi_0 - \dfrac{\log \alpha}{n(2\alpha+\beta)} \text{ or } t < \xi_0 . \\ 0, & \text{otherwise.} \end{cases}$

Then, $E_{\xi_0}\{\Phi^*(X, Y)\}$

$$= P_{\xi_0}\left\{T > \xi_0 - \frac{\log \alpha}{n(2\alpha+\beta)}\right\} + P_{\xi_0}(T < \xi_0).$$

$$= E_{\xi_0}\{\Phi_2^*(X, Y)\} = \alpha.$$

For $\xi < \xi_0$,

$$\beta_{\Phi^*} = P_\xi\left\{T > \xi_0 - \frac{\log \alpha}{n(2\alpha+\beta)}\right\} + P_\xi(T < \xi_0)$$

$$= P_\xi\{n(2\alpha+\beta)(T-\xi) > n(2\alpha+\beta)(\xi_0 - \xi) - \log\alpha\}$$

$$+ P_\xi\{n(2\alpha+\beta)(T-\xi) < n(2\alpha+\beta)(\xi_0 - \xi)\}$$

$$= \exp\{-n(2\alpha+\beta)(\xi_0-\xi)+\log\alpha\}$$

$$+ 1 - \exp\{-n(2\alpha+\beta)(\xi_0-\xi)\}$$

$$= 1-(1-\alpha)\exp\{-n(2\alpha+\beta)(\xi_0-\xi)\};\ \xi < \xi_0.$$

$$\beta_{\Phi_1^*}(\xi) = P_\xi\left\{T < \xi_0 - \frac{\log(1-\alpha)}{n(2\alpha+\beta)}\right\}$$

$$= P_\xi\{n(2\alpha+\beta)(T-\xi) < n(2\alpha+\beta)(\xi_0-\xi) - \log(1-\alpha)\}$$

$$= 1-(1-\alpha)\ \exp\{-n(2\alpha+\beta)(\xi_0-\xi)\}$$

$$= \beta_{\Phi^*}(\xi);\ \xi < \xi_0.$$

For $\xi > \xi_0$,

$$\beta_{\Phi^*} = P_\xi\left\{T > \xi_0 - \frac{\log\alpha}{n(2\alpha+\beta)}\right\} + P_\xi(T < \xi_0)$$

$$= P_\xi\left\{T > \xi_0 - \frac{\log\alpha}{n(2\alpha+\beta)}\right\}$$

$$= \beta_{\Phi_2^*}(\xi);\ \xi > \xi_0.$$

Therefore,

$$\beta_{\Phi^*}(\xi) = \begin{cases} \beta_{\Phi_1^*}(\xi), & \xi < \xi_0 . \\ \beta_{\Phi_2^*}(\xi), & \xi > \xi_0 . \end{cases}$$

Also $\beta_{\Phi^*}(\xi) > \beta_{\Phi}(\xi)$ for every level $\alpha$ test $\Phi$, since $\Phi_1^*$ and $\Phi_2^*$ are UMP level $\alpha$ tests for $\xi < \xi_0$ and $\xi > \xi_0$ respectively. Therefore, $\Phi^*$ is the UMP level $\alpha$ test for testing H: $\xi = \xi_0$ vs K: $\xi \neq \xi_0$ .

**Remark 1.5.1**

The above arguments are similar to those available for **E** $(\xi, 1)$, $\xi \in R$ in Lehmann (1986).

# ABSOLUTELY CONTINUOUS BIVARIATE EXPONENTIAL

# SCALE MODEL

## 2.0 Summary

In this chapter we consider a scale family arising out of ACBVE with equal marginals. Lehmann and Casella (1998) provide a detailed discussion on scale family and equivariant estimation of scale parameter. In this chapter we discuss the problem of equivariant estimation of the scale parameter of ACBVE model belonging to a scale family. Some distributional results also have been presented.

Section 2.1 provides some definitions and notations required in this chapter. Some distributional results on ACBVE are discussed in Section 2.2. In Section 2.3 we obtained the UMVUE of the scale parameter. Section 2.4 deals with the equivariant estimation of the scale parameter of an ACBVE belonging to a scale family. The Pitman estimator is also derived. Some particular cases are discussed. In Section 2.5 we derive optimal tests for hypothesis concerning the scale parameter.

## 2.1 Preliminaries

Let X be a random variable (vector) with the distribution function

$F_\tau(.)$, $\tau \in R$. Let $\{F_\tau, \tau \in R\}$ be a scale family so that $F_\tau(x) = F(x/\tau)$ for some distribution function F.

In this chapter we are interested in the equivariant estimation of the scale parameter τ. Following Lehmann and Casella (1998) minimum risk equivariant estimator (MREE) of τ is given by

$$\delta^*(x) = \frac{\delta_0(x)}{w^*(z)}$$

where $\delta_0$ is a scale equivariant estimator, $w^*$ minimizes $E_1\left\{\rho\left(\frac{\delta_0}{w(\mathbf{z})}\right) \middle| \mathbf{Z} = \mathbf{z}\right\}$ with respect to w and the loss function is

$$L(\tau; \delta) = \left(\frac{\delta}{\tau} - 1\right)^2 .$$

Here $\mathbf{z} = (z_1, z_2, \ldots, z_n)$ where $z_i = x_i / g(\mathbf{x})$, $i = 1, 2, 3, \ldots, n$. Further g is such that $g > 0$ and $g(b\mathbf{x}) = b\, g(\mathbf{x})$, for every $b > 0$.

From Definition 1.1.1 we have the pdf of ACBVE with equal marginals is,

$$f(x, y) = \frac{(\alpha+\beta)(2\alpha+\beta)}{2} \exp[-\alpha(x+y)-\beta(x \vee y)],$$

$$x, y > 0; \quad \alpha, \beta > 0.$$

The following pdf provides a scale family, for fixed (α, β

Let

$$f_\tau(x, y) = \left\{\frac{(\alpha+\beta)(2\alpha+\beta)}{2\tau^2}\right\} \exp\left[-\frac{1}{\tau}\{\alpha(x+y)-\beta(x \vee y)\}\right],$$

$x, y > 0; \tau > 0 \ldots$ (2.1.1).

For fixed (α, β), the distribution of (X/τ, Y/τ) does not depend on τ. Therefore, when (α, β) is known; the family in (2.1.1) is a scale family with the scale parameter τ. The transformation $(X^*, Y^*) = (bX, bY)$ induces the transformation $\tau^* = b\tau$ on the parameter space and the model is invariant.

## 2.2 Distributional Results

Given a random variable from (2.1.1), we wish to find the distribution of

$$T = \sum_{i=1}^{n} \{\alpha(X_i + Y_i) + \beta(X_i \vee Y_i)\}.$$

**Theorem 2.2.1** The distribution of $2T/\tau$ is Chi-square with 4n d.f.

**Proof** Let m (t) be the mgf of T at t.

Then $m(t) = \exp\left[t \sum_{i=1}^{n} \{\alpha(x_i + y_i) + \beta(x_i \vee y_i)\}\right]$

$$= [\text{mgf of } \{\alpha(x_1 + y_1) + \beta(x_1 \vee y_1)\}]^n$$

$$= \{m_1(t)\}^n, \text{ say.} \qquad \dots (2.2.1)$$

Now $m_1(t)$

$$= \int_0^\infty \int_0^\infty \left\{\frac{(\alpha+\beta)(2\alpha+\beta)}{2\tau^2}\right\} \exp[t\{\alpha(x+y) + \beta(x \vee y)\}] \exp\left[-\left(\frac{1}{\tau}\right)\{\alpha(x+y) + \beta(x \vee y)\}\right] dx\, dy$$

$$= \left\{\frac{(\alpha+\beta)(2\alpha+\beta)}{2\tau^2}\right\} \int_0^\infty \int_0^\infty \exp\left[-\left(\frac{1}{\tau} - t\right)\{\alpha(x+y) + \beta(x \vee y)\}\right] dx\, dy$$

$$= \left\{\frac{(\alpha+\beta)(2\alpha+\beta)}{2\tau^2}\right\} \left\{\frac{2}{(\alpha+\beta)(2\alpha+\beta)\left(\frac{1}{\tau} - t\right)^2}\right\}$$

$= (1 - t\tau)^{-2}$.

Hence $m(t) = (1 - t\tau)^{-2n}$, in view of (2.2.1).

Hence $2T/\tau$ has the Chi-square distribution with $4n$ d.f.

## 2.3 UMVUE

We wish to find the UMVUE of $\tau^r$.

Consider

$$\delta_0(\mathbf{x}, \mathbf{y}) = \left[\sum_{i=1}^{n} \{\alpha(x_i + y_i) + \beta(x_i \vee y_i)\}\right]^r .$$

From Theorem 2.2.1

$$T \sim G(1/\tau, 2n).$$

It follows that

$$E_\tau\{\delta_0(X, Y)\} = E_\tau(T^r)$$

$$= \frac{\Gamma(2n+r)}{\Gamma 2n} \tau^r .$$

Thus, $E_\tau\left\{\frac{\Gamma 2n}{\Gamma(2n+r)}\delta_0\right\} = \tau^r$ for every $\tau$.

Since $T$ is complete sufficient, $\frac{\Gamma 2n}{\Gamma(2n+r)}\delta_0$ is the UMVUE of $\tau^r$. In particular, the UMVUE of $\tau$ is $T/(2n)$.

## Maximum Likelihood Estimator

The likelihood function is

$L(\tau \quad \mathbf{x,y})$

$$=\left\{\frac{(\alpha+\beta)(2\alpha+\beta)}{2\tau^2}\right\}^n \exp\left[-\frac{1}{\tau}\sum_{i=1}^{n}\{\alpha(x_i+y_i)+\beta(x_i\vee y_i)\}\right],$$

$x_i, y_i > 0$, for every i. … (2.3.1)

Thus $\log L = k - 2n\log\tau - (1/\tau)\,t$.

$$\frac{\partial\log L}{\partial\tau} = -\,(2n/\tau) + (t/\tau^2).$$

$$\hat{\tau} = (t/2n).$$

$$\left.\frac{\partial^2\log L}{\partial\tau^2}\right|_{\tau=\hat{\tau}} < 0.$$

Now $E_\tau(\hat{\tau}) = (1/2n)\,E_\tau(T)$

$= (1/2n)\,2n\,\tau$

$= \tau$ for every $\tau$.

$2\hat{\tau} = (1/n)\sum_i U_i$, where $U_i^{\text{'s}}$ are iid $G(1/\tau, 2)$ random variables. Further $E_\tau(U_1) = 2\tau$ and $V_\tau(U_1) = 2\tau^2$. By Lindeberg – Levy Central Limit Theorem,

$$\frac{\sqrt{n}\,(2\hat{\tau}-2\tau)}{\sqrt{2}\,\tau} \xrightarrow{d} N(0,1).$$

That is $\sqrt{n}\,(\hat{\tau}-\tau) \xrightarrow{d} N(0,\tau^2/2)$ for every τ.

Hence MLE is CAN for estimating τ with the asymptotic variance $\tau^{2}/(2n)$.

## 2.4 Equivariant Estimation

Assuming that (α, β) is known, we wish to find the MREE of $\tau^r$. Let $(X_i, Y_i)$, $i = 1, 2, 3, \ldots, n$ be a random sample from a distribution with the pdf given in (2.1.1) so that the joint pdf is given by (2.3.1). Clearly the distribution of (**X**, **Y**) belongs to a scale family with τ as the scale parameter. We are interested in estimating $\tau^r$*, , following Lehmann and Casella (1998).

Take $\delta_0(\mathbf{x}, \mathbf{y}) = \left[\sum_{i=1}^{n} \{\alpha\,(x_i + y_i) + \beta\,(x_i \vee y_i)\}\right]^r$.

Consider $\delta_0(b\mathbf{x}, b\mathbf{y}) = \left[\sum_{i=1}^{n} \{\alpha\,(bx_i + by_i) + \beta\,(bx_i \vee by_i)\}\right]^r$

$= b^r \left[\sum_{i=1}^{n} \{\alpha\,(x_i + y_i) + \beta\,(x_i \vee y_i)\}\right]^r$

$= b^r\,\delta_0(\mathbf{x}, \mathbf{y})$.

*, $\tau^r$ is τ to the power r.

Thus $\delta_0$ **(x, y)** is a scale equivariant estimator for estimating $\tau^r$.

From Section 2.2

$$T = \sum_{i=1}^{n} \{\alpha(X_i + Y_i) + \beta(X_i \vee Y_i)\} \sim G?1/\tau,\ 2n?.$$

It is clear that T is a complete sufficient statistic and so $\delta_0$ **(x, y)** is also complete sufficient.

When $\tau = 1$,

$$T \sim G?1,\ 2n?.$$

Then $E_1\ \{\delta_0\ (\mathbf{X}, \mathbf{Y})\} = \dfrac{\Gamma(2n+r)}{\Gamma 2n}$ and $E_1\ \{\delta_0^2\ (\mathbf{X}, \mathbf{Y})\}$

$$= \frac{\Gamma(2n+2r)}{\Gamma 2n}.$$

Therefore, MREE of $\tau^r$ with respect to the loss function $L?\tau;\ \delta? = \{(\delta/\tau) - 1\}^2$ is

$$\delta^*?\mathbf{X}, \mathbf{Y}? = \frac{\Gamma(2n+r)}{\Gamma(2n+2r)} \left[\sum_{i=1}^{n} \{\alpha\,(X_i + Y_i) + \beta\,(X_i \vee Y_i)\}\right]^r.$$

**Pitman Form**

When the loss function is $L?\tau^r;\ \delta? = \{(\delta/\tau^r) - 1\}^2$, the MREE of $\tau^r$ is given by

$$\delta^*(\mathbf{x}) = \frac{\int_0^\infty \upsilon^{n+r-1} f(\upsilon x_1, \upsilon x_2, \ldots, \upsilon x_n)\, d\upsilon}{\int_0^\infty \upsilon^{n+2r-1} f(\upsilon x_1, \upsilon x_2, \ldots, \upsilon x_n)\, d\upsilon}.$$

**Proof** $\delta^*(\mathbf{x}) = \delta_0(\mathbf{x}) \dfrac{E_1\{\delta_0(\mathbf{X}) | \mathbf{z}\}}{E_1\{\delta_0^2(\mathbf{X}) | \mathbf{z}\}}$.

Let $\delta_0(\mathbf{X}) = X_n^r$. To compute $E_1\{\delta_0(\mathbf{X})\, \mathbf{z}\}$, make the change of variables, $z_i = \dfrac{x_i}{|x_n|}$, $i = 1, 2, 3, \ldots, (n-1)$; $z_n = x_n$.

**Case (i): $x_n > 0$**

In this case we have $z_i = \dfrac{x_i}{x_n}$, $i = 1, 2, 3, \ldots, (n-1)$; $z_n = x_n$.

The Jacobian of the transformation is $J = z_n^{n-1}$.

The joint density of $(z_1, z_2, \ldots, z_{n-1})$ is therefore

$$p_\tau(z_1, z_2, \ldots, z_{n-1}) = z_n^{n-1} f(z_1 z_n, z_2 z_n, \ldots, z_{n-1} z_n, z_n).$$

and the conditional density of $Z_n$ given $\mathbf{z} = (z_1, z_2, \ldots, z_{n-1})$ is,

$$\frac{z_n^{n-1} f(z_1 z_n, z_2 z_n, \ldots, z_{n-1} z_n, z_n)}{\int t^{n-1} f(z_1 t, z_2 t, \ldots, z_{n-1} t, t)\, dt}.$$

It follows that,

$$E_1\{\delta_0(\mathbf{X})|\mathbf{z}\} = E_1\{Z_n^r|\mathbf{z}\}$$

$$= \frac{\int t^{n+r-1} f(z_1 t, z_2 t, \ldots, z_{n-1} t, t)\,dt}{\int t^{n-1} f(z_1 t, z_2 t, \ldots, z_{n-1} t, t)\,dt}.$$

This can be expressed in terms of x's as

$$E_1\{X_n^r|\mathbf{z}\} = \frac{\int_0^\infty t^{n+r-1} f\left(\frac{x_1}{x_n}t, \frac{x_2}{x_n}t, \ldots, \frac{x_{n-1}}{x_n}t, t\right)dt}{\int_0^\infty t^{n-1} f\left(\frac{x_1}{x_n}t, \frac{x_2}{x_n}t, \ldots, \frac{x_{n-1}}{x_n}t, t\right)dt}.$$

Take $\upsilon = t/x_n$. Then $dt = x_n\, d\upsilon$.

Therefore, $E_1\{X_n^r|\mathbf{z}\} =$

$$\frac{\int_0^\infty (\upsilon x_n)^{n+r-1} x_n f(\upsilon x_1, \upsilon x_2, \ldots, \upsilon x_n)\,d\upsilon}{\int_0^\infty (\upsilon x_n)^{n-1} x_n f(\upsilon x_1, \upsilon x_2, \ldots, \upsilon x_n)\,d\upsilon}$$

$$= x_n^r \frac{\int_0^\infty \upsilon^{n+r-1} f(\upsilon x_1, \upsilon x_2, \ldots, \upsilon x_n)\,d\upsilon}{\int_0^\infty \upsilon^{n-1} f(\upsilon x_1, \upsilon x_2, \ldots, \upsilon x_n)\,d\upsilon}.$$

Similarly, we have

$$E_1\{\delta_0^2(\mathbf{X})|\mathbf{z}\} = E_1\{X_n^{2r}|\mathbf{z}\} =$$

$$x_n^{2r}\frac{\int_0^\infty \upsilon^{n+2r-1} f(\upsilon x_1, \upsilon x_2, \ldots, \upsilon x_n)d\upsilon}{\int_0^\infty \upsilon^{n-1} f(\upsilon x_1, \upsilon x_2, \ldots, \upsilon x_n)d\upsilon}.$$

Therefore,

$$\frac{E_1\{\delta_0(\mathbf{X})|\mathbf{z}\}}{E_1\{\delta_0^2(\mathbf{X})|\mathbf{z}\}} = \frac{1}{\delta_0(\mathbf{x})}\frac{\int_0^\infty \upsilon^{n+r-1} f(\upsilon x_1, \upsilon x_2, \ldots, \upsilon x_n)d\upsilon}{\int_0^\infty \upsilon^{n+2r-1} f(\upsilon x_1, \upsilon x_2, \ldots, \upsilon x_n)d\upsilon}.$$

Thus $\delta^*(\mathbf{x}) = \dfrac{\int_0^\infty \upsilon^{n+r-1} f(\upsilon x_1, \upsilon x_2, \ldots, \upsilon x_n)d\upsilon}{\int_0^\infty \upsilon^{n+2r-1} f(\upsilon x_1, \upsilon x_2, \ldots, \upsilon x_n)d\upsilon}.$

**Case (ii):** $x_n < 0$

In this case we have $z_i = -\dfrac{x_i}{x_n}$, $i = 1, 2, 3, \ldots, (n-1)$; $z_n = -x$

The Jacobian of the transformation is $J = -z_n^{n-1}$

Then by the above arguments we have

$$\delta^*(\mathbf{x}) = \frac{\int_0^\infty \upsilon^{n+r-1} f(\upsilon x_1, \upsilon x_2, \ldots, \upsilon x_n)d\upsilon}{\int_0^\infty \upsilon^{n+2r-1} f(\upsilon x_1, \upsilon x_2, \ldots, \upsilon x_n)d\upsilon}.$$

Lehmann and Casella (1998) provide the Pitman form for the MREE of the scale parameter with respect to the squared error loss as follows

$$\delta^*(\mathbf{x}) = \frac{\int_0^\infty \upsilon^{n+r-1} f(\upsilon x_1, \upsilon x_2, \ldots, \upsilon x_n)d\upsilon}{\int_0^\infty \upsilon^{n+2r-1} f(\upsilon x_1, \upsilon x_2, \ldots, \upsilon x_n)d\upsilon}.$$

Then the Pitman form for the bivariate sample is

$$\delta^*(\mathbf{x}, \mathbf{y}) =$$

$$\frac{\int_0^\infty \upsilon^{2n+r-1} f(\upsilon x_1, \upsilon y_1, \upsilon x_2, \upsilon y_2, \ldots, \upsilon x_n \upsilon y_n)d\upsilon}{\int_0^\infty \upsilon^{2n+2r-1} f(\upsilon x_1, \upsilon y_1, \upsilon x_2, \upsilon y_2, \ldots, \upsilon x_n \upsilon y_n)d\upsilon}$$

$$= \frac{\int_0^\infty \upsilon^{2n+r-1} \exp\left[-\upsilon \sum_{i=1}^{n} \{\alpha(x_i + y_i) + \beta(x_i \vee y_i))\}\right] d\upsilon}{\int_0^\infty \upsilon^{2n+2r-1} \exp\left[-\upsilon \sum_{i=1}^{n} \{\alpha(x_i + y_i) + \beta(x_i \vee y_i))\}\right] d\upsilon}$$

$$= \frac{\Gamma(2n+r)}{\delta_0^{(2n+r-1)/r}} \; \frac{\delta_0^{(2n+2r-1)/r}}{\Gamma(2n+2r)}$$

$$= \frac{\Gamma(2n+r)}{\Gamma(2n+2r)} \delta_0 .$$

That is, $\delta^*(x, y) = \delta_0\,(x, y) \dfrac{\Gamma(2n+r)}{\Gamma(2n+2r)}$. In particular MREE of $\tau$ is $T/(2n+1)$.

**Particular cases**

(i) Suppose that $\alpha = 1, \beta = 1$

Then MREE of $\tau^r$ is given by

$$\delta^*(X, Y) = \delta_0\,(X, Y) \frac{\Gamma(2n+r)}{\Gamma(2n+2r)} .$$

(ii) Suppose that $\alpha = 1, \beta = 2$

Then MREE of $\tau^r$ is given by

$$\delta^*(X, Y) = \delta_0\,(X, Y) \frac{\Gamma(2n+r)}{\Gamma(2n+2r)} .$$

(iii) Suppose that $\alpha = 2, \beta = 1$

Then MREE of $\tau^r$ is given by

$$\delta^*(X, Y) = \delta_0\,(X, Y) \frac{\Gamma(2n+r)}{\Gamma(2n+2r)} .$$

### MREE with respect to absolute error loss function

If the loss function is of the form $L(\tau^r; \delta) = \frac{|\delta - \tau^r|}{\tau^r}$, the MREE of $\tau^r$ is given by $\delta^*(x, y) = \frac{\delta_0(x, y)}{w^*(z)}$, where $\delta_0$ (**x**, **y**) is a scale equivariant estimator and $w^*$ is any scale median of $\delta_0$ (**x**, **y**) for $\tau = 1$.

In our problem we have

$\delta_0(\mathbf{x},\mathbf{y}) = \left[\sum_{i=1}^{n} \{\alpha\,(x_i + y_i) + \beta\,(x_i \vee y_i)\}\right]^r$ and $w^*$ is such that

$$\frac{\int_0^{w^*} z_{2n}^{2n} f(z_1 z_{2n}, z_2 z_{2n}, \ldots, z_{2n-1} z_{2n}, z_{2n})\, dz_{2n}}{\int_0^{\infty} t^{2n-1} f(z_1 t, z_2 t, \ldots, z_{2n-1} t, t)\, dt}$$

$$= \frac{\int_{w^*}^{\infty} z_{2n}^{2n} f(z_1 z_{2n}, z_2 z_{2n}, \ldots, z_{2n-1} z_{2n}, z_{2n})\, dz_{2n}}{\int_0^{\infty} t^{2n-1} f(z_1 t, z_2 t, \ldots, z_{2n-1} t, t)\, dt}$$

It follows that,

$$\int_0^{w^*} \left\{\frac{(\alpha+\beta)(2\alpha+\beta)}{2}\right\}^n z_{2n}^{2n} \exp\left[-\sum_{i=1}^{n} \{\alpha\,(x_i + y_i) + \beta\,(x_i \vee y_i)\}\right] dz_{2n}$$

$$= \int_{w^*}^{\infty} \left\{\frac{(\alpha+\beta)(2\alpha+\beta)}{2}\right\}^n z_{2n}^{2n} \exp\left[-\sum_{i=1}^{n} \{\alpha\,(x_i + y_i) + \beta\,(x_i \vee y_i)\}\right] dz_{2n}$$

Take $z_{2n} = y_n \upsilon$, then we have,

$$= \int_{w^*}^{\infty} (y_n \upsilon)^{2n} y_n \exp\left[-\upsilon \sum_{i=1}^{n} \{\alpha (x_i + y_i) + \beta (x_i \vee y_i)\}\right] d\upsilon.$$

This implies,

$$= \int_{w^*}^{\infty} (\upsilon)^{2n} y_n \exp\left[-\upsilon \sum_{i=1}^{n} \{\alpha (x_i + y_i) + \beta (x_i \vee y_i)\}\right] d\upsilon.$$

By using incomplete Gamma tables we can get the value of $w^*$.

## 2.5 UMP Tests

We are interested in testing

$H: \tau \leq \tau_0$ vs. $K: \tau > \tau_0$.

We have,

$$p_\tau(\mathbf{x},\mathbf{y}) = \left(\frac{(\alpha+\beta)(2\alpha+\beta)}{2\tau^2}\right)^n \exp\left[-\frac{1}{\tau} \sum_{i=1}^{n} \{\alpha(x_i + y_i) + \beta(x_i \vee y_i)\}\right],$$

$x_i, y_i > 0, \tau > 0.$

$= C(\tau) \exp\{Q(\tau)\, T(\mathbf{x}, \mathbf{y})\}\, h(\mathbf{x}, \mathbf{y})$, where C

$$(\tau) = \left(\frac{(\alpha+\beta)(2\alpha+\beta)}{2\tau^2}\right)^n, \quad Q(\tau) = -\frac{1}{\tau}$$

increasing in $\tau$, if $\tau > \tau_0$, $T(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{n}\{\alpha(x_i + y_i) + \beta(x_i \vee y_i)\}$ and $h(\mathbf{x}, \mathbf{y}) = 1$. Hence the family $\{P_\tau, \tau > 0\}$ possesses MLR property in $T(X, Y)$. Therefore, the UMP level $\alpha$ test for testing $H: \tau \leq \tau_0$ vs. $K: \tau > \tau_0$ is given by (Lehmann, 1986)

$$\Phi(t) = \begin{cases} 1, & \text{if } (2t/\tau_0) > c. \\ 0, & \text{otherwise.} \end{cases}$$

where c can be determined using Chi-square table with 4n degrees of freedom, in view of Theorem 2.2.1.

**ark 2.5.1** UMP level $\alpha$ test can be obtained for testing $H: \tau \geq \tau_0$ vs. $K: \tau < \tau_0$ in a similar way.

# ABSOLUTELY CONTINUOUS BIVARIATE EXPONENTIAL

## LOCATION-SCALE MODEL

### 3.0 Summary

In this chapter we consider a location-scale family arising out of Absolutely Continuous Bivariate Exponential (ACBVE) distribution with equal marginals. Edwin Prabakaran and Chandrasekar (1994) proposed simultaneous equivariant estimation of the location – scale parameter. In this chapter we discuss the problem of simultaneous equivariant estimation of the location – scale parameter of an ACBVE model belonging to a location-scale family. Some distributional results also have been presented.

Section 3.1 provides some definitions and notations required in this chapter. Some distributional results on ACBVE are discussed in Section 3.2. In Section 3.3 we discuss the estimation of UMRUE of the location-scale parameter. Section 3.4 deals with the simultaneous equivariant estimation of the location–scale parameter of an ACBVE model belonging to a location-scale family. Some particular cases are also discussed. In Section 3.5 we discuss optimal estimation of the percentile of the population.

## 3.1 Preliminaries

Let X be a random variable (vector) with the distribution function $F_{\xi,\tau}(.)$, $\xi \in R$, $\tau > 0$. Let $\{F_{\xi,\tau}$, $\xi \in R$, $\tau > 0\}$ be a location-scale family so that $F_{\xi,\tau} = F\left(\frac{x-\xi}{\tau}\right)$ for some distribution function F. Here the probability density function f corresponding to F is

f(**u**=

$$\frac{(\alpha+\beta)(2\alpha+\beta)}{2}\exp\{-\alpha(u_1+u_2)-\beta(u_1 \vee u_2)\}, u_1, u_2 > 0; \quad \alpha,$$

β known.

In this chapter we are interested in the simultaneous equivariant estimation of the location-scale parameter (ξ, τ)'. Following Edwin Prabakaran and Chandrasekar (1994), MREE of (ξ, τ)' with respect to quadratic type loss function

$$L(\theta; \delta) = a_{11}\left(\frac{\delta_1-\xi}{\tau}\right)^2 + 2a_{12}\left(\frac{\delta_1-\xi}{\tau}\right)\left(\frac{\delta_2}{\tau}-1\right) +$$

$$a_{22}\left(\frac{\delta_2}{\tau}-1\right)^2$$

is given by $\delta^{**} = (\delta_1^{**}, \delta_2^{**})$ where $\delta_1^{**}(\mathbf{x}, \mathbf{y}) = \delta_{01}(\mathbf{x}, \mathbf{y}) - g(\mathbf{x}, \mathbf{y})\, w_1^*(\mathbf{z})$ and $\delta_2^{**}(\mathbf{x}, \mathbf{y}) = \delta_{02}(\mathbf{x}, \mathbf{y}) / w_2^*(\mathbf{z})$. Here $\delta_0 = (\delta_{01}, \delta_{02})$ is a fixed equivariant estimator.

Also $g(a + b\mathbf{x}, a + b\mathbf{y}) = bg(\mathbf{x}, \mathbf{y}),\ g > 0$,

$$w_1^*(\mathbf{z}) = \frac{E_{01}(\delta_{01}\delta_{02} \mid \mathbf{z})}{E_{01}(\delta^2_{02} \mid \mathbf{z})},$$

$$w_2^*(\mathbf{z}) = \frac{E_{01}(\delta^2_{02} \mid \mathbf{z})}{E_{01}(\delta_{02} \mid \mathbf{z})} \text{ and}$$

$$z_i = \frac{(x_i - x_n)}{g},\ i = 1, 2, 3, \ldots, (n-1).$$

The following pdf provides a location – scale family

$$f_{\xi,\tau}(x, y) =$$

$$\frac{(\alpha+\beta)(2\alpha+\beta)}{2\tau^2} \exp\left[-\frac{1}{\tau}\{\alpha(x+y) + \beta(x \vee y) - (2\alpha+\beta)\xi\}\right],$$

$$x, y > \xi,\ \xi \in R,\ \tau > 0. \qquad \ldots(3.1.1)$$

For fixed $(\alpha, \beta)$, the distribution of $\left(\frac{X-\xi}{\tau}, \frac{Y-\xi}{\tau}\right)$ does not depend on $(\xi, \tau)'$. Therefore the family in (3.1.1) is a location-scale family with the location-scale parameter $(\xi, \tau)'$. The transformation $(X^*, Y^*) = (a + bX, a + bY)$

induces the transformation $(\xi^*, \tau^*) = (a + b\xi, b\tau)$ on the parameter space and the model is invariant.

## 3.2 Distributional Results

The following theorem is valid for any $\alpha, \beta > 0$.

**Theorem 3.2.1** If $(X, Y) \sim$ ACBVE $(\alpha, \beta)$, then

$$(X-Y) \overset{d}{=} \frac{1}{2}F_1 + \frac{1}{2}F_2,$$ where

$F_1(x) = 1 - \exp\left\{-\left(\frac{\alpha+\beta}{\tau}x\right)\right\}$, $x > 0$ and $F_2(x) = \exp\left(\frac{\alpha+\beta}{\tau}x\right)$, $x < 0$.

**Proof** Let (X, Y) be ACBVE with the pdf given in (3.1.1). We wish to find the distribution of $(X - Y)$. The MGF of $(X - Y)$ at t is given by

$E[\exp\{t(X-Y)\}] =$

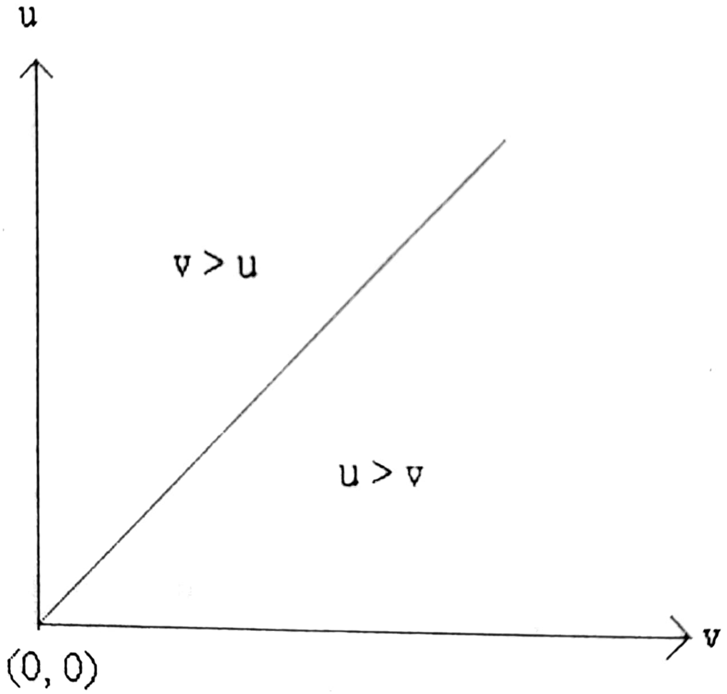
$$\int_{\xi}^{\infty}\int_{\xi}^{\infty} \frac{(\alpha+\beta)(2\alpha+\beta)}{2\tau^2} \exp\{t(x-y)\}$$

$$\exp\left[-\left(\frac{1}{\tau}\right)\{\alpha(x+y) + \beta(x \vee y) - (2\alpha+\beta)\xi\}\right] dx\, dy$$

Consider the transformation, $u = \frac{x-\xi}{\tau}$ and $v = \frac{y-\xi}{\tau}$. The Jacobian of the transformation is $\tau^2$. Then the MGF is

$$E[\exp\{t\tau(U-V)\}]=$$

$$\int_0^\infty\int_0^\infty \frac{(\alpha+\beta)(2\alpha+\beta)}{2}\exp\{\tau t\,(u-v)-\alpha(u+v)-\beta(u\vee v)\}\,dvdu$$



$$=\int_0^\infty\left[\int_0^u \frac{(\alpha+\beta)(2\alpha+\beta)}{2}\exp\{-(\alpha+\beta-\tau t)u-(\alpha+\tau t)v\}dv\right]du$$

$$+\int_0^\infty\left[\int_u^\infty \frac{(\alpha+\beta)(2\alpha+\beta)}{2}\exp\{-(\alpha-\tau t)u-(\alpha+\beta+\tau t)v\}dv\right]du$$

$$= \frac{(\alpha+\beta)(2\alpha+\beta)}{2}\int_0^{\infty}\frac{1}{(\alpha+\tau t)}\exp\{-(\alpha+\beta-\tau t)u\}$$

$$[1-\exp\{-(\alpha+\tau t)u\}]du$$

$$+ \frac{(\alpha+\beta)(2\alpha+\beta)}{2}\int_0^{\infty}\frac{1}{(\alpha+\beta+\tau t)}\exp\{-(\alpha-\tau t)u\}$$

$$\exp\{-(\alpha+\beta+\tau t)u\}du$$

$$= \frac{(\alpha+\beta)(2\alpha+\beta)}{2}\left\{\frac{1}{(\alpha+\tau t)(\alpha+\beta-\tau t)} - \frac{1}{(\alpha+\tau t)(2\alpha+\beta)} + \frac{1}{(\alpha+\beta+\tau t)(2\alpha+\beta)}\right\}$$

$$= \frac{(\alpha+\beta)(2\alpha+\beta)}{2}\left\{\frac{(\alpha+\beta+\tau t)(2\alpha+\beta)-(\alpha+\beta+\tau t)(\alpha+\beta-\tau t)+(\alpha+\tau t)(\alpha+\beta-\tau t)}{(\alpha+\tau t)(\alpha+\beta-\tau t)(\alpha+\beta+\tau t)(2\alpha-\beta)}\right\}$$

$$= (\alpha+\beta)\left\{\frac{(\alpha+\beta)}{(\alpha+\beta-\tau t)(\alpha+\beta+\tau t)}\right\}$$

$$= \frac{(\alpha+\beta)^2}{(\alpha+\beta)^2-(\tau t)^2}$$

$$= \left\{1-\left(\frac{\tau t}{\alpha+\beta}\right)^2\right\}^{-1}$$

$$= \left(1-\frac{\tau t}{\alpha+\beta}\right)^{-1}\left(1+\frac{\tau t}{\alpha+\beta}\right)^{-1}$$

$=\frac{1}{2}\left(1-\frac{\tau t}{\alpha+\beta}\right)^{-1}+\frac{1}{2}\left(1+\frac{\tau t}{\alpha+\beta}\right)^{-1}$, Since a + b = 2 implies that $\frac{1}{ab}=\frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}\right)$.

Thus, $(X-Y)\overset{d}{=}\frac{1}{2}F_1+\frac{1}{2}F_2$, where

$F_1(x) = 1-\exp\left\{-\left(\frac{\alpha+\beta}{\tau}x\right)\right\}$, $x > 0$ and

$F_2(x)=\exp\left(\frac{\alpha+\beta}{\tau}x\right)$, $x<0$.

**Theorem 3.2.2** If (X, Y) ~ ACBVE (α, β), then $|X-Y| \sim E\left(0,\frac{\tau}{\alpha+\beta}\right)$.

**Proof** Let $F^*$ be the distribution function of $|X-Y|$.

For $u > 0$,

$$\begin{aligned}F^*(u) &= P\,(|X-Y| \le u)\\ &= P\,\{-u \le (X-Y) \le u\}\\ &= P\,\{(X-Y) \le u\} - P\,\{(X-Y) \le -u\}\\ &= F\,(u) - F\,(-u).\end{aligned}$$

From Theorem 3.2.1, we have,

$$F(u) = 1 - \frac{1}{2}\exp\left\{-\left(\frac{\alpha+\beta}{\tau}u\right)\right\},\ u > 0 \text{ and}$$

$$F(-u) = \frac{1}{2}\exp\left\{-\left(\frac{\alpha+\beta}{\tau}u\right)\right\},\ u < 0.$$

Therefore, $F^{*}(u) = 1 - \frac{1}{2}\exp\left\{-\left(\frac{\alpha+\beta}{\tau}u\right)\right\} - \frac{1}{2}\exp\left\{-\left(\frac{\alpha+\beta}{\tau}u\right)\right\}$

$$= 1 - \exp\left\{-\left(\frac{\alpha+\beta}{\tau}u\right)\right\}.$$

Hence the theorem.

The following theorem is similar to the one in Block and Basu (1974), stated without proof.

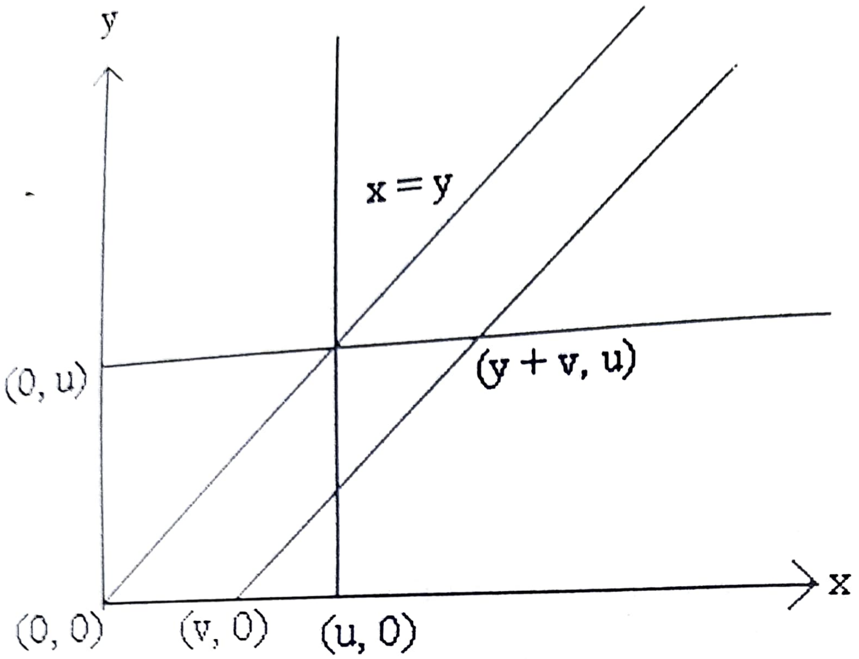
**Theorem 3.2.3** If (X, Y) ~ ACBVE ($\alpha$, $\beta$), then $(X \wedge Y)$ and $(X - Y)$ are independent.

**Proof** Let G and F be the distribution functions of $(X \wedge Y)$ and $(X - Y)$ respectively.

Let $\overline{H}(u, v) = P\{(X \wedge Y) > u;\ (X - Y) > v\}$.

For $u > 0$, $v > 0$;

$$\overline{H}(u, v) = P\{X > u,\ Y > u;\ (X - Y) > v\}$$



It follows that,

$$\overline{H}(u, v) = \int_{u}^{\infty}\left[\int_{y+v}^{\infty} \frac{(\alpha+\beta)(2\alpha+\beta)}{2\tau^{2}} \exp\left\{-\frac{\alpha}{\tau}y - \frac{(\alpha+\beta)}{\tau}x + \frac{(2\alpha+\beta)}{\tau}\xi\right\} dx\right] dy$$

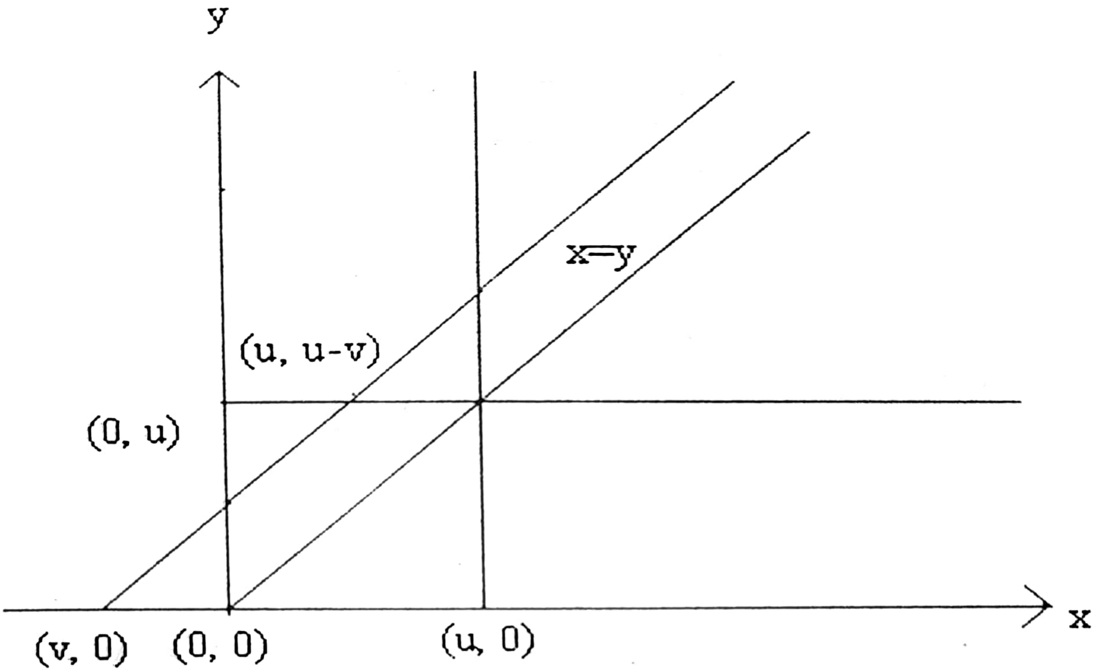
$$= \int_{u}^{\infty} \frac{(\alpha+\beta)(2\alpha+\beta)}{2\tau^{2}} \frac{\tau}{(\alpha+\beta)} \exp\left\{-\frac{\alpha}{\tau}y - \frac{(\alpha+\beta)}{\tau}(y+v) + \frac{(2\alpha+\beta)}{\tau}\xi\right\} dy$$

$$= \frac{(\alpha+\beta)(2\alpha+\beta)}{2\tau^{2}} \frac{\tau}{(\alpha+\beta)} \frac{\tau}{(2\alpha+\beta)} \exp\left\{-\frac{\alpha+\beta}{\tau}v - \frac{(2\alpha+\beta)}{\tau}u + \frac{(2\alpha+\beta)}{\tau}\xi\right\}$$

$$= \frac{1}{2}\exp\left\{-\frac{1}{\tau}(2\alpha+\beta)(u-\xi) - \frac{1}{\tau}(\alpha+\beta)v\right\}$$

$$= \overline{G}(u)\overline{F}(v) .$$

For $u > 0,\ v < 0$,



$$\bar{H}(u, v) = P\ \{X > u,\ Y > u;\ (X - Y) > v\}$$

$$= P\ \{X > u,\ Y > u\} - P\ \{X > u,\ Y > u;\ (X - Y) \le v\}$$

$$= P\ \{(X \wedge Y) > u\} -$$

$$\int_{u-v}^{\infty}\int_{u}^{y+v} \frac{(\alpha+\beta)(2\alpha+\beta)}{2\tau^2} \exp\left[-\frac{1}{\tau}\{\alpha x + (\alpha+\beta)y - (2\alpha+\beta)\xi\}\right] dx\,dy$$

$$= \exp\left\{-\frac{(2\alpha+\beta)}{\tau}(u - \xi)\right\} - \frac{(\alpha+\beta)(2\alpha+\beta)}{2\tau^2} \exp\left\{\frac{(2\alpha+\beta)}{\tau}\xi\right\}$$

$$\left[\int_{u-v}^{\infty} \frac{\tau}{\alpha} \exp\left(-\frac{(\alpha+\beta)}{\tau} y\right)\left\{\exp\left(-\frac{\alpha}{\tau} u\right) - \exp\left(-\frac{(2\alpha+\beta)}{\tau}(y+v)\right)\right\}\right]$$

$$= \exp\left\{-\frac{1}{\tau}(2\alpha+\beta)(u-\xi)\right\} - \frac{1}{2}\exp\left[-\frac{1}{\tau}\{(\alpha+\beta)v + (2\alpha+\beta)(u-\xi)\}\right]$$

$$= \exp\left\{-\frac{1}{\tau}(2\alpha+\beta)(u-\xi)\right\}\left[1 - \frac{1}{2}\exp\left\{-\frac{(\alpha+\beta)}{\tau} v\right\}\right]$$

$$= \overline{G}(u)\,\overline{F}(v).$$

Therefore, $\overline{H}(u, v) = \overline{G}(u)\,\overline{F}(v)$ ; $u > 0$, $v < 0$.

For $u < 0$,

$$\overline{H}(u, v) = \overline{G}(u) ; \quad \overline{F}(v) = 1.$$

Hence $\overline{H}(u, v) = \overline{G}(u)\,\overline{F}(v)$ for every $u, v \in R$.

Thus $(X \wedge Y)$ is independent of $(X - Y)$.

**Remark 3.2.1**

Since $(X - Y)$ is independent of $(X \wedge Y)$, $|X - Y|$ is also independent of $(X \wedge Y)$.

## Sufficient Statistic

Let $(X_i, Y_i)$, $i = 1, 2, 3, \ldots, n$ be a random sample from (3.1.1).

The join pdf of $(X_i, Y_i)$, $i = 1, 2, 3, \ldots, n$ is,

$p(\mathbf{x}, \mathbf{y}; \xi, \tau) =$

$$\left\{\frac{(\alpha+\beta)(2\alpha+\beta)}{2\tau^2}\right\}^n \exp\left[-\frac{1}{\tau}\sum_{i=1}^{n}\{\alpha(x_i+y_i)+\beta(x_i\vee y_i)-(2\alpha+\beta)\xi\}\right],$$

$$\operatorname*{Min}_i(X_i \wedge Y_i) > \xi.$$

Let $U_i = X_i \wedge Y_i$, $i = 1, 2, 3, \ldots, n$.

Then, $p(\mathbf{x}, \mathbf{y}; \xi, \tau)$

$$\left\{\frac{(\alpha+\beta)(2\alpha+\beta)}{2\tau^2}\right\}^n \exp\left[-\frac{1}{\tau}\sum_{i=1}^{n}\left\{\alpha(x_i+y_i)+\beta(x_i\vee y_i)-(2\alpha+\beta)U_{(1)}\right\}\right]$$

$$\exp\left\{-\frac{n(2\alpha+\beta)}{\tau}(U_{(1)}-\xi)\right\}$$

$= g_{\xi,\tau}(t_1, t_2)\ h(\mathbf{x}, \mathbf{y})$, where $T_1 = U_{(1)}$, $T_2$

$$= \sum_{i=1}^{n}\{\alpha(x_i+y_i)+\beta(x_i\vee y_i)-(2\alpha+\beta)U_{(1)}\},$$

$g_{\xi,\tau}(t_1,t_2)=$

$$\left\{\frac{(\alpha+\beta)(2\alpha+\beta)}{2\tau^2}\right\}^n \exp\left[-\frac{1}{\tau}\{t_2+n(2\alpha+\beta)(t_1-\xi)\}\right]$$

and $h(\mathbf{x}, \mathbf{y}) = 1$.

By Factorization theorem $(T_1, T_2)$ is a sufficient statistic.

**Theorem 3.2.4**

(i) $T_1 \sim E\left(\xi, \dfrac{\tau}{n(2\alpha+\beta)}\right)$,

(ii) $T_2 \sim G\left(\dfrac{1}{\tau}, 2n-1\right)$ and

(iii) $T_1 \coprod T_2$.

**Proof**

(i) $U_1, U_2, \ldots, U_n$ are iid $E\left(\xi, \dfrac{\tau}{(2\alpha+\beta)}\right)$ random variables. This implies that

(ii) $T_1 = U_{(1)} \sim E\left(\xi, \dfrac{\tau}{n(2\alpha+\beta)}\right)$.

(iii) For $i = 1, 2, 3, \ldots, n$

$$\alpha(x_i + y_i) + \beta(x_i \vee y_i) = \alpha(x_i + y_i) + \frac{\beta}{2}\{(x_i + y_i) + |x_i - y_i|\}$$

$$= \frac{(2\alpha+\beta)}{2}(x_i + y_i) + \frac{\beta}{2}|x_i - y_i|$$

$$= \frac{(2\alpha+\beta)}{2}\{2(x_i \wedge y_i) + |x_i - y_i|\} + \frac{\beta}{2}|x_i - y_i|$$

$$= (2\alpha + \beta)(x_i \wedge y_i) + (\alpha + \beta)|x_i - y_i|$$

$$= (2\alpha + \beta)\, u_i + (\alpha + \beta)|x_i - y_i|$$

Therefore,

$$T_2 = \sum_{i=1}^{n} \{(2\alpha + \beta)\, u_i + (\alpha + \beta)|x_i - y_i| - (2\alpha + \beta)\, u_{(1)}\}$$

$$= (2\alpha + \beta) \sum_{i=1}^{n} (u_i - u_{(1)}) + (\alpha + \beta) \sum_{i=1}^{n} |x_i - y_i|$$

$$= (2\alpha + \beta) \sum_{i=2}^{n} (u_{(i)} - u_{(1)}) + (\alpha + \beta) \sum_{i=1}^{n} |x_i - y_i|. \qquad \ldots (3.2.1)$$

Now $u_i \overset{\text{iid}}{\sim} E\left(\xi, \dfrac{\tau}{(2\alpha+\beta)}\right)$ or $(2\alpha + \beta)\, u_i \sim E(\xi, \tau)$.

From the result on spacings from exponential distribution (Sukhatme, 1937), we have

$$(2\alpha + \beta) \sum_{i=2}^{n} (u_{(i)} - u_{(1)}) \sim G\left(\frac{1}{\tau}, n-1\right). \qquad \ldots (3.2.2)$$

$(2\alpha + \beta)\, U_{(1)} \sim E\left(\xi, \dfrac{\tau}{n}\right)$ and $U_{(1)} \amalg \sum_{i=2}^{n} (u_{(i)} - u_{(1)})$.

From Theorem 3.2.2, we have $|X_i - Y_i| \sim E\left(0, \frac{\tau}{(\alpha+\beta)}\right)$.

This implies $(\alpha+\beta)\sum_{i=1}^{n}|X_i - Y_i| \sim G\left(\frac{1}{\tau}, n\right)$.

... (3.2.3)

Further, from Theorem 3.2.3 $U_{(i)} \perp\!\!\!\perp |X_i - Y_i|$.

Therefore, (3.2.1), (3.2.2) and (3.2.3)

implies that the MGF of $T_2$ at t is

$$m(t) = (1 - t\tau)^{-(n-1)} (1 - t\tau)^{-n}$$

$$= (1 - t\tau)^{-(2n-1)}.$$

Therefore, $T_2 \sim G\left(\frac{1}{\tau}, 2n-1\right)$.

(i) For fixed $\tau$, $T_1 = U_{(1)}$ is complete sufficient and $T_2$ is ancilliary. Then by

Basu`s theorem $T_1 \perp\!\!\!\perp T_2$.

## 3.3 UMRUE

Following Lehmann and Casella (1998), it can be shown that $(T_1, T_2)$ is complete. Let $\theta = (\xi, \tau)'$. $E_\theta(T_1)$

$= \left\{ \xi + \dfrac{\tau}{n(2\ \alpha + \beta)} \right\}$, $E_\theta\ (T_2) = (2n\text{-}1)\tau$. Therefore

$E_\theta\left( \dfrac{T_2}{2n-1} \right) = \tau$ and $E_\theta \left\{ T_1 - \dfrac{T_2}{n(2n-1)(2\alpha+\beta)} \right\} = \xi$, for every $\theta$.

Thus $\delta_1^* = \left\{ T_1 - \dfrac{T_2}{n(2n-1)(2\alpha+\beta)} \right\}$ and

$\delta_2^* = \dfrac{T_2}{(2n-1)}$ are the UMVUE's of $\xi$ and $\tau$ respectively. Following Lehmann and Casella (1998) $\delta^* = (\delta_1^*, \delta_2^*)$ is the UMRUE of $(\xi, \tau)'$ with respect to any loss function which is convex in the second argument. From Kale and Chandrasekar (1983) and Chandrasekar (1984) $\delta^*$ is M-optimal, D-optimal, T-optimal and $Q_A$-optimal.

## 3.4 Equivariant Estimation

Let $(X_i, Y_i)$, $i = 1, 2, 3, \ldots, n$ be a random sample from a distribution with pdf given in (3.1.1). Clearly the distribution of **(X, Y)** belongs to a location–scale family with $(\xi, \tau)'$ as the location-scale parameter. We are interested in estimating $(\xi, \tau)'$ following Edwin Prabakaran and Chandrasekar (1994).

Take $\delta_{01} = T_1$ and $\delta_{02} = T_2$. Then $\delta_0 = (\delta_{01}, \delta_{02})'$.

We have $\delta_{01}(a + b\mathbf{x}, a + b\mathbf{y}) = a + b\,\delta_{01}(\mathbf{x}, \mathbf{y})$ and $\delta_{02}(a + b\mathbf{x}, a + b\mathbf{y}) = b\,\delta_{02}(\mathbf{x}, \mathbf{y})$. Thus $\delta_0 = (\delta_{01}, \delta_{02})'$ is a complete sufficient location-scale equivariant estimator.

When $(\xi, \tau)' = (0, 1)'$,

$$T_1 \sim E\left(0, \frac{1}{n(2\alpha+\beta)}\right), \quad T_2 \sim G(1, 2n-1),$$

$$w_1^* = \frac{(2n-1)}{n(2\alpha+\beta)\, 2n\,(2n-1)}$$

$$= \frac{1}{2n^2(2\alpha+\beta)},$$

$$\text{and } w_2^* = \frac{2n(2n-1)}{(2n-1)}$$

$$= 2n.$$

$$\text{Therefore } \delta_1^{**} = \delta_{01} - \frac{\delta_{02}}{2n^2(2\alpha+\beta)} \text{ and } \delta_2^{**} = \frac{\delta_{02}}{2n}.$$

## 3.5 Equivariant estimation of percentiles

We are interested in estimating $\eta = c\xi + d\tau$; $c, d \in R$. From Datta and Ghosh (1988), $\delta$ is said to be an equivariant estimator for $\eta$ if,

$$\delta(a + b\mathbf{x}) = c\,a + b\,\delta(\mathbf{x}), \text{ for every } a \in R,\ b > 0.$$

For details regarding MREE for percentiles, the reader is referred to Datta and Ghosh (1988) and Amala Revathy (2003).

Then the MREE of $\eta$ is given by

$$\delta^*(\mathbf{x}) = \delta_0(\mathbf{x}) - g(\mathbf{x})\, w^*(\mathbf{z})$$

where $w^*$ minimizes $E_{01}\left[\rho\{(\delta_0(\mathbf{x}) - g(\mathbf{x})\, w(\mathbf{z})) | \mathbf{z}\}\right]$ with respect to w and $\mathbf{z}$.

$z_i = \dfrac{(x_i - x_n)}{g(\mathbf{x})}$, where $g(a + b\mathbf{x}) = b\, g(\mathbf{x})$, $g > 0$.

When loss is of the form $(\delta - d)^2$ and $g = \delta_{02}$,

$$w^* = \frac{E_{01}\{(\delta - d)\, g | \mathbf{z}\}}{E_{01}\{g^2 | \mathbf{z}\}}.$$

In our problem, consider

$\delta_{01}(\mathbf{x,y}) = \underset{i}{\mathrm{Min}}(x_i \wedge y_i)$ and $\delta_{01}(\mathbf{x,y})$

$$= \sum_{i=1}^{n} \left\{\alpha(x_i + y_i) + \beta(x_i \vee y_i) - (2\alpha + \beta)\underset{i}{\mathrm{Min}}(x_i \wedge y_i)\right\}.$$

Clearly, $(\delta_{01}, \delta_{02})'$ is complete sufficient and $g = \delta_{02} > 0$.

Take $\delta_0 = c\delta_{01} + d\delta_{02}$.

Then $\delta_0(a + b\mathbf{x}) = c\delta_{01}(a + b\mathbf{x}) + d\delta_{02}(a + b\mathbf{x})$

$$= c\ \{a + b\delta_{01}(\mathbf{x})\} + d\ b\delta_{02}(\mathbf{x})$$

$$= c\,a + b\{c\delta_{01}(\mathbf{x}) + d\delta_{02}(\mathbf{x})\}$$

$$= c\,a + b\delta_0(\mathbf{x})$$

This implies that $\delta_0$ is a location-scale equivariant estimator.

Also we have $\delta_{01} \sim E\left(\xi, \frac{\tau}{n(2\alpha+\beta)}\right)$ and $\delta_{02} \sim G\left(\frac{1}{\tau}, 2n-1\right)$.

Therefore $w^* = \dfrac{E_{01}\{(\delta_0 - d)\,\delta_{02}|\mathbf{z}\}}{E_{01}(\delta_{02}^2|\mathbf{z})}$

$$= \frac{E_{01}\{(c\delta_{01} + d\delta_{02} - d)\,\delta_{02}\}}{E_{01}(\delta_{02}^2)}$$

$$= \frac{cE_{01}(\delta_{01})\,E_{01}(\delta_{02}) + dE_{01}(\delta_{02}^2) - dE_{01}(\delta_{02})}{E_{01}(\delta_{02}^2)}$$

$$= \frac{c\dfrac{1}{n(2\alpha+\beta)}(2n-1) + d\,2n\,(2n-1) - d(2n-1)}{2n\,(2n-1)}$$

$$= \frac{(2n-1)\left\{\dfrac{c}{n(2\alpha+\beta)} + d(2n-1)\right\}}{2n\,(2n-1)}$$

$$= \frac{1}{2n}\left\{\frac{c}{n(2\alpha+\beta)} + d(2n-1)\right\}$$

Therefore the MREE of $\eta$ is,

$$\delta^{*}(\mathbf{X}, \mathbf{Y}) = c\,\delta_{01} + \frac{1}{2n}\left\{d - \frac{c}{n(2\alpha+\beta)}\right\}\delta_{02}$$

**Remark 3.5.1** By taking c = 1 and d = 0, we obtain the MREE of $\xi$ as

$$\delta_1^{*}(\mathbf{X}, \mathbf{Y}) = \delta_{01} - \frac{1}{2n^2(2\alpha+\beta)}\delta_{02}.$$

taking c = 0 and d = 1, we obtain the MREE of $\tau$ as

$$\delta_2^{*}(\mathbf{X}, \mathbf{Y}) = \frac{1}{2n}\delta_{02}.$$

These expressions coincide with the MREE of $\xi$ and $\tau$ obtained in the previous section.

# RELIABILITY MEASURES OF SYSTEMS WITH ACBVE COMPONENTS

## 4.0 Summary

This chapter deals with reliability aspects of systems arising out of ACBVE components. In this chapter we consider a two-component system when failure times follow ACBVE distribution with equal marginals. We obtain the reliability performance measures of the two component systems. Also we discuss the estimation of these reliability performance measures

Section 4.1 provides some definitions and notations required in this chapter. Some distributional results are discussed in Section 4.2. In Section 4.3 we obtain the reliability performance measures of two-component parallel, series and standby systems when the component failure times follow ACBVE with equal marginals. Section 4.4 deals with the estimation of reliability performance measures.

## 4.1 Preliminaries

Consider a two component system with failure times $T_1$ and $T_2$ respectively. Assume that $(T_1, T_2)$ follows ACBVE $(\alpha, \beta, \xi, \tau)$ with pdf

$$f(t_1, t_2; \alpha, \beta, \xi, \tau) =$$

$$\frac{(\alpha+\beta)(2\alpha+\beta)}{2\tau^2} \exp\left[-\frac{1}{\tau}\{\alpha(t_1+t_2)+\beta(t_1 \vee t_2)-(2\alpha+\beta)\xi\}\right],$$

$$\alpha, \beta \text{ fixed}, \xi \in R, \tau > 0; t_1 \wedge t_2 > \xi. \quad \ldots.(4.1.1)$$

It is assumed that $(\alpha, \beta)$ is known.

Suppose we observe n identical systems with observations $(t_{1i}, t_{2i})$, $i = 1, 2, 3, \ldots, n$. Then the joint pdf of the sample is

$$p_{\xi,\tau}(\mathbf{t_1}, \mathbf{t_2}) =$$

$$\left\{\frac{(\alpha+\beta)(2\alpha+\beta)}{2\tau^2}\right\}^n \exp\left[-\frac{1}{\tau}\sum_i\{\alpha(t_{1i}+t_{2i})+\beta(t_{1i} \vee t_{2i})-(2\alpha+\beta)\xi\}\right],$$

$$\operatorname*{Min}_i(t_{1i} \wedge t_{2i}) > \xi; \ \xi \in R, \tau > 0.$$

It is easy to observe that the maximum likelihood estimator (MLE) of $(\xi, \tau)$ is given by $(\hat{\xi}, \hat{\tau})$, where

$\hat{\xi} = \operatorname*{Min}_i(t_{1i} \wedge t_{2i})$ and

$$\hat{\tau} = \frac{1}{n}\sum_i\left\{\alpha(t_{1i}+t_{2i})+\beta(t_{1i} \vee t_{2i})-(2\alpha+\beta)\hat{\xi}\right\}.$$

## 4.2 Distributional results

**Theorem 4.2.1** Let $(T_1, T_2)$ follows ACBVE $(\alpha, \beta, \xi, \tau)$ with pdf given in (4.1.1). Then

(i) $T_1 + T_2 - 2\xi \overset{d}{=} U_1 + U_2$,

(ii) $T_1 \vee T_2 - \xi \overset{d}{=} U_1 + U_3$ and

(iii) $T_1 \wedge T_2 - \xi \overset{d}{=} U_3$,

where $U_1 \sim E\left(0, \dfrac{\tau}{(\alpha+\beta)}\right)$, $U_2 \sim E\left(0, \dfrac{2\tau}{(2\alpha+\beta)}\right)$ and $U_3 \sim$

$E\left(0, \dfrac{\tau}{(2\alpha+\beta)}\right)$.

**Proof** MGF of $(T_1 + T_2, T_1 \vee T_2)$ at $(u_1, u_2)$ is

$$m(u_1, u_2) =$$

$$\frac{(\alpha+\beta)(2\alpha+\beta)}{2\tau^2} \int\int \exp\left[-\frac{1}{\tau}\left\{\begin{matrix}(\alpha - u_1\tau)(t_1 + t_2) + \\ (\beta - u_2\tau)(t_1 \vee t_2) - (2\alpha+\beta)\xi\end{matrix}\right\}\right] dt_1\, dt_2$$

$$= \frac{(\alpha+\beta)(2\alpha+\beta)}{2\tau^2} \frac{2\tau^2 \exp\left[\left\{\left(\dfrac{2\alpha+\beta}{\tau}\right)\xi - \dfrac{2(\alpha-u_1\tau)+(\beta-u_2\tau)}{\tau}\xi\right\}\Big/\tau\right]}{\{(\alpha-u_1\tau)+(\beta-u_2\tau)\}\{2(\alpha-u_1\tau)+(\beta-u_2\tau)\}}$$

$$= \left(1-\frac{u_1+u_2}{(\alpha+\beta)}\tau\right)^{-1}\left(1-\frac{2u_1+u_2}{(2\alpha+\beta)}\tau\right)^{-1}\exp\{(2u_1+u_2)\xi\}.$$

$$\text{(i)}\, m(u_1,0) = \left(1-\frac{u_1}{(\alpha+\beta)}\tau\right)^{-1}\left(1-\frac{2u_1}{(2\alpha+\beta)}\tau\right)^{-1}\exp(2u_1\xi).$$

This implies that $T_1 + T_2 - 2\xi \overset{d}{=} U_1 + U_2$, where $U_1 \sim E\left(0,\frac{\tau}{(\alpha+\beta)}\right)$, $U_2 \sim E\left(0,\frac{2\tau}{(2\alpha+\beta)}\right)$ and $U_1 \amalg U_2$.

$$\text{(ii)} \quad m(0,u_2) \left(1-\frac{u_2}{(\alpha+\beta)}\tau\right)^{-1}\left(1-\frac{u_2}{(2\alpha+\beta)}\tau\right)^{-1}\exp(u_2\,\xi).$$

This implies that $T_1 \vee T_2 - \xi \overset{d}{=} U_1 + U_3$, where $U_1 \sim E\left(0,\frac{\tau}{(\alpha+\beta)}\right)$, $U_3 \sim E\left(0,\frac{\tau}{(2\alpha+\beta)}\right)$ and $U_1 \amalg U_3$.

(iii) From Section 1.2, we have,

$$T_1 \wedge T_2 - \xi \overset{d}{=} U_3, \text{ where } U_3 \sim E\left(0,\frac{\tau}{(2\alpha+\beta)}\right).$$

## 4.3 Reliability performance measures

### 4.3.1 Parallel system

**MTBF** Consider a two-unit parallel system with component failure times $T_1$ and $T_2$ respectively. Then the system failure time is,

$$T = T_1 \vee T_2.$$

Assume that $(T_1, T_2) \sim$ ACBVE $(\alpha, \beta, \xi, \tau)$. Then from Theorem 4.2.1, we have

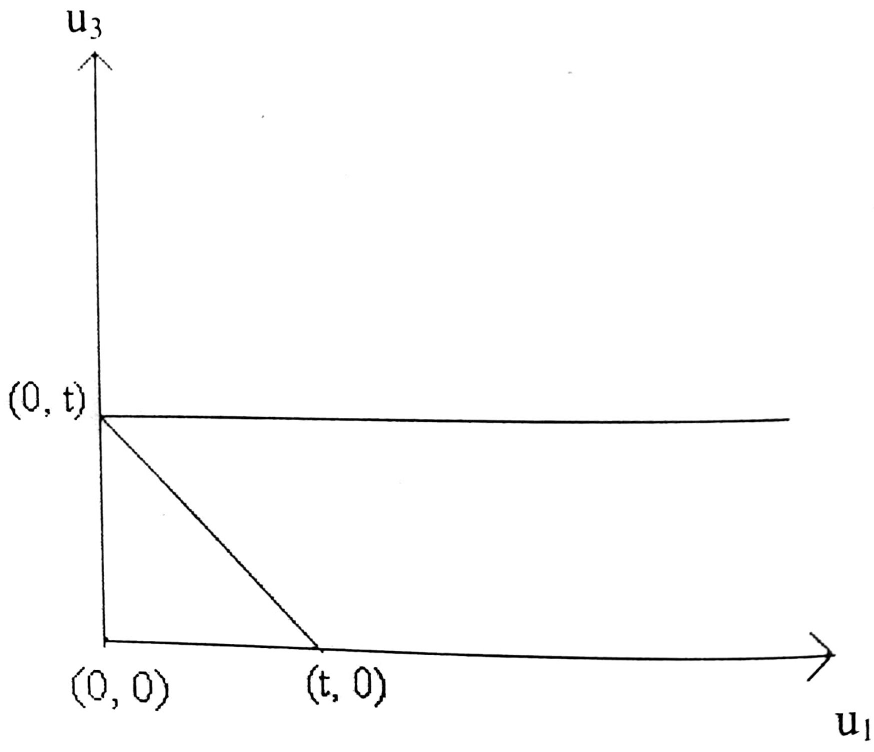
$$T - \xi \overset{d}{=} U_1 + U_3,$$

where $U_1 \sim E\left(0, \frac{\tau}{(\alpha+\beta)}\right)$ and $U_3 \sim E\left(0, \frac{\tau}{(2\alpha+\beta)}\right)$.

$$\begin{aligned}\text{Therefore, MTBF} &= E(T) \\ &= E(U_1 + U_3) + \xi \\ &= \frac{\tau}{(\alpha+\beta)} + \frac{\tau}{(2\alpha+\beta)} + \xi \\ &= \frac{3\alpha+2\beta}{(\alpha+\beta)(2\alpha+\beta)}\tau + \xi.\end{aligned}$$

**Reliability function** Consider, $P(T - \xi > t), t > 0.$

$$= P(U_1 + U_3 > t), t > 0.$$



$$= \int_{0}^{t}\int_{t-u_3}^{\infty} \frac{(\alpha+\beta)(2\alpha+\beta)}{\tau^2} \exp\left\{-\left(\frac{\alpha+\beta}{\tau}\right)u_1 - \left(\frac{2\alpha+\beta}{\tau}\right)u_3\right\} du_1 du_3$$

$$+ \int_{t}^{\infty}\int_{0}^{\infty} \frac{(\alpha+\beta)(2\alpha+\beta)}{\tau^2} \exp\left\{-\left(\frac{\alpha+\beta}{\tau}\right)u_1 - \left(\frac{2\alpha+\beta}{\tau}\right)u_3\right\} du_1 du_3$$

$$= \int_{0}^{t} \frac{(2\alpha+\beta)}{\tau} \exp\left\{-\left(\frac{\alpha+\beta}{\tau}\right)(t-u_3) - \left(\frac{2\alpha+\beta}{\tau}\right)u_3\right\} du_3\, u_1$$

$$+ \int_{t}^{\infty} \frac{(2\alpha+\beta)}{\tau} \exp\left\{-\left(\frac{2\alpha+\beta}{\tau}\right)u_3\right\} du_3$$

$$= \left(\frac{2\alpha+\beta}{\alpha}\right) \exp\left\{-\left(\frac{\alpha+\beta}{\tau}\right)t\right\}\left\{1-\exp\left(-\frac{\alpha}{\tau}t\right)\right\} + \exp\left\{-\left(\frac{2\alpha+\beta}{\tau}\right)t\right\}$$

$$\left(\frac{2\alpha+\beta}{\alpha}\right)\left[\exp\left\{-\left(\frac{\alpha+\beta}{\tau}\right)t\right\} - \exp\left\{-\left(\frac{2\alpha+\beta}{\tau}\right)t\right\}\right] + \exp\left\{-\left(\frac{2\alpha+\beta}{\tau}\right)t\right\}$$

$$\left(\frac{2\alpha+2\beta}{\alpha}\right)\exp\left\{-\left(\frac{\alpha+\beta}{\tau}\right)t\right\} - \left(\frac{\alpha+\beta}{\alpha}\right)\exp\left\{-\left(\frac{2\alpha+\beta}{\tau}\right)t\right\},\ t > 0.$$

Therefore,

$P\ (T - \xi > t - \xi)$

$$\left(\frac{2\alpha+\beta}{\alpha}\right)\exp\left\{-\left(\frac{\alpha+\beta}{\tau}\right)(t-\xi)\right\} - \left(\frac{\alpha+\beta}{\alpha}\right)\exp\left\{-\left(\frac{2\alpha+\beta}{\tau}\right)(t-\xi)\right\},\ t > \xi.$$

Then, the reliability function is given by

$$R\ (t) = P(T > t) =$$

$$\left(\frac{2\alpha+\beta}{\alpha}\right)\exp\left\{-\left(\frac{\alpha+\beta}{\tau}\right)(t-\xi)\right\}-\left(\frac{\alpha+\beta}{\alpha}\right)\exp\left\{-\left(\frac{2\alpha+\beta}{\tau}\right)(t-\xi)\right\},\ t>\xi.$$

### 4.3.1 Series system

**MTBF**

Consider a two unit series system with component failure times $T_1$ and $T_2$ respectively. Then the system failure time is

$$T = T_1 \wedge T_2.$$

Assume that $(T_1, T_2) \sim$ ACBVE $(\alpha, \beta, \xi, \tau)$. Then from Theorem 4.2.1, we have

$$T - \xi \overset{d}{=} U_3 \quad \text{where } U_3 \sim E\left(0, \frac{\tau}{(2\alpha+\beta)}\right).$$

Therefore, MTBF $= E\ (T)$

$$= E\ (U_3) + \xi$$

$$= \frac{\tau}{2\alpha+\beta} + \xi.$$

**Reliability function** Consider, $P\ (T - \xi > t),\ t > 0$

$$= P\ (U_3 > t),\ t > 0.$$

$$= \int_t^{\infty} \left(\frac{2\alpha+\beta}{\tau}\right) \exp\left\{-\left(\frac{2\alpha+\beta}{\tau}\right)u_3\right\} du_3$$

$$= \exp\left\{-\left(\frac{2\alpha+\beta}{\tau}\right)t\right\},\ t > 0.$$

Therefore,

$$P(T-\xi > t - \xi) = \exp\left\{-\left(\frac{2\alpha+\beta}{\tau}\right)(t-\xi)\right\},\ t > \xi.$$

Then, the reliability function is given by

$$R\,(t) = P\,(T > t) = \exp\left\{-\left(\frac{2\alpha+\beta}{\tau}\right)(t-\xi)\right\},\ t > \xi.$$

### 4.3.1 Standby system

#### MTBF

Consider a two unit standby system with component failure times $T_1$ and $T_2$ respectively. Then the system failure time is

$$T = T_1 + T_2.$$

Assume that $(T_1, T_2) \sim$ ACBVE $(\alpha, \beta, \xi, \tau)$. Then from Theorem 4.2.1, we have

$$T - 2\xi \overset{d}{=} U_1 + U_2$$

where $U_1 \sim E\left(0, \frac{\tau}{(\alpha+\beta)}\right)$ and $U_2 \sim E\left(0, \frac{2\tau}{(2\alpha+\beta)}\right)$.

Therefore, MTBF = E (T)

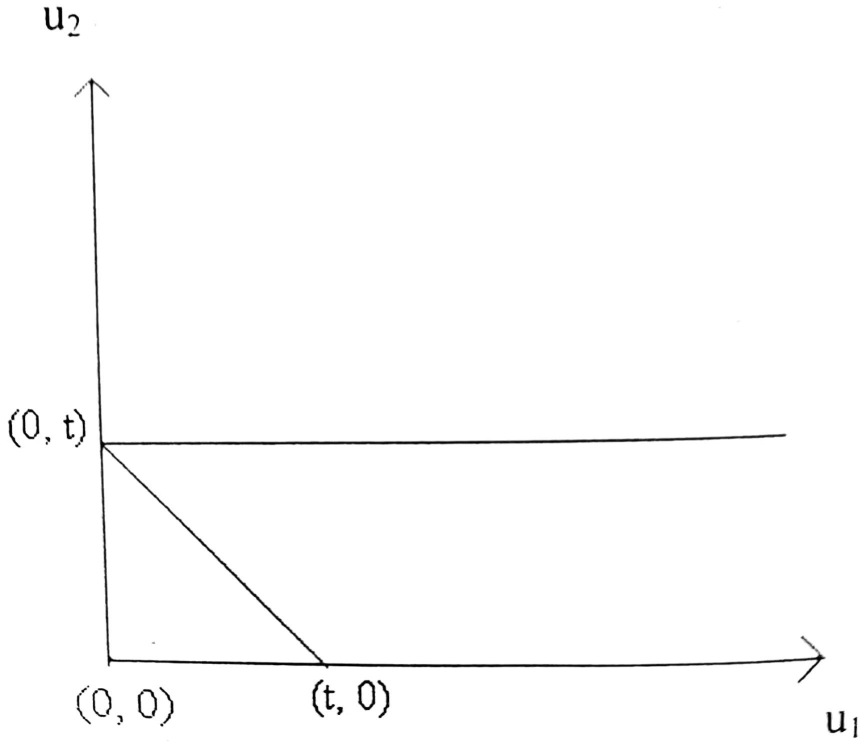
$$= E\,(U_1 + U_2) + 2\xi$$

$$= \frac{\tau}{\alpha+\beta} + \frac{2\tau}{2\alpha+\beta} + 2\xi$$

$$= \frac{4\alpha+3\beta}{(\alpha+\beta)(2\alpha+\beta)}\tau + 2\xi.$$

**Reliability function** Consider, $P\,(T - 2\xi > t),\ t > 0.$

$$= P\,(U_1 + U_2 > t),\ t > 0.$$



$$= \int_0^t \int_{t-u_2}^{\infty} \frac{(\alpha+\beta)(2\alpha+\beta)}{2\tau^2} \exp\left\{-\left(\frac{\alpha+\beta}{\tau}\right)u_1 - \left(\frac{2\alpha+\beta}{2\tau}\right)u_2\right\} du_1 du_2$$

$$+ \int_t^{\infty} \int_0^{\infty} \frac{(\alpha+\beta)(2\alpha+\beta)}{2\tau^2} \exp\left\{-\left(\frac{\alpha+\beta}{\tau}\right)u_1 - \left(\frac{2\alpha+\beta}{2\tau}\right)u_2\right\} du_1 du_2$$

$$= \int_0^t \frac{(2\alpha+\beta)}{2\tau} \exp\left\{-\left(\frac{\alpha+\beta}{\tau}\right)(t-u_2) - \left(\frac{2\alpha+\beta}{2\tau}\right)u_2\right\} du_2$$

$$+ \int_t^{\infty} \frac{(2\alpha+\beta)}{2\tau} \exp\left\{-\left(\frac{2\alpha+\beta}{2\tau}\right)u_2\right\} du_2$$

$$= \left(\frac{2\alpha+\beta}{\beta}\right) \exp\left\{-\left(\frac{\alpha+\beta}{\tau}\right)t\right\}\left\{\exp\left(\frac{\beta}{2\tau}t\right) - 1\right\} + \exp\left\{-\left(\frac{2\alpha+\beta}{2\tau}\right)t\right\}$$

$$\left(\frac{2\alpha+\beta}{\beta}\right)\left[\exp\left\{-\left(\frac{2\alpha+\beta}{2\tau}\right)t\right\} - \exp\left\{-\left(\frac{\alpha+\beta}{\tau}\right)t\right\}\right] + \exp\left\{-\left(\frac{2\alpha+\beta}{2\tau}\right)t\right\}$$

$$\left(\frac{2\alpha+2\beta}{\beta}\right)\exp\left\{-\left(\frac{2\alpha+\beta}{2\tau}\right)t\right\} - \left(\frac{2\alpha+\beta}{\beta}\right)\exp\left\{-\left(\frac{\alpha+\beta}{\tau}\right)t\right\},\ t > 0.$$

Therefore,

$P\,(T - 2\xi > t - 2\xi) =$

$$\left(\frac{2\alpha+2\beta}{\beta}\right)\exp\left\{-\left(\frac{2\alpha+\beta}{2\tau}\right)(t-2\xi)\right\} - \left(\frac{2\alpha+\beta}{\beta}\right)\exp\left\{-\left(\frac{\alpha+\beta}{\tau}\right)(t-2\xi)\right\},\ t > 2\xi.$$

Then, the reliability function is given by

R(t)=P(T>t)=

$$\left(\frac{2\alpha+2\beta}{\beta}\right)\exp\left\{-\left(\frac{2\alpha+\beta}{2\tau}\right)(t-2\xi)\right\}-\left(\frac{2\alpha+\beta}{\beta}\right)\exp\left\{-\left(\frac{\alpha+\beta}{\tau}\right)(t-2\xi)\right\},\ t>2\xi.$$

## 4.4 Estimation of reliability measures

We discuss optimal estimation of reliability measures for the three systems considered in this chapter.

### 4.4.1 Parallel system

**UMVUE** The MTBF is

$$M=\frac{3\alpha+2\beta}{(\alpha+\beta)(2\alpha+\beta)}\tau+\xi\,.$$

From Section 3.3, the UMVUEs of $\xi$ and $\tau$ are

$$\left\{T_1-\frac{T_2}{n(2n-1)(2\alpha+\beta)}\right\} \text{ and } \frac{T_2}{(2n-1)}$$ respectively. Hence

the UMVUE of M is

$$M^*=\left\{\frac{3\alpha+2\beta}{(\alpha+\beta)(2\alpha+\beta)}\right\}\left(\frac{T_2}{2n-1}\right)+\left\{T_1-\frac{T_2}{n(2n-1)(2\alpha+\beta)}\right\}.$$

$$=T_1+\left\{\frac{(3n-1)\alpha+(2n-1)\beta}{n(2n-1)(\alpha+\beta)(2\alpha+\beta)}\right\}T_2.$$

**MREE** From Section 3.4, the MREE of $c\xi + d\tau$ is

$$M^{**}=c\,T_1+\frac{1}{2n}\left\{d-\frac{c}{n(2\alpha+\beta)}\right\}T_2.$$

Here $c=1$, $d=\left\{\dfrac{3\alpha+2\beta}{(\alpha+\beta)(2\alpha+\beta)}\right\}$.

Then $M^{**}=T_1+\left\{\dfrac{(3n-1)\alpha+(2n-1)\beta}{2n^2(\alpha+\beta)(2\alpha+\beta)}\right\}T_2.$

**MLE** The MLE of $(\xi, \tau)$ is

$$\hat{\xi}=T_1 \text{ and } \hat{\tau}=\frac{1}{2n}T_2.$$

Then the MLE of MTBF is

$$\hat{M}=\frac{3\alpha+2\beta}{(\alpha+\beta)(2\alpha+\beta)}\hat{\tau}+\hat{\xi}.$$

The MLE of the reliability function is

$$\hat{R}(t)=\left(\frac{2\alpha+\beta}{\alpha}\right)\exp\left\{-\left(\frac{\alpha+\beta}{\hat{\tau}}\right)(t-\hat{\xi})\right\}$$

$$-\left(\frac{\alpha+\beta}{\alpha}\right)\exp\left\{-\left(\frac{2\alpha+\beta}{\hat{\tau}}\right)(t-\hat{\xi})\right\},\ t>\hat{\xi}.$$

### 4.4.1 Series system

**UMVUE** The MTBF is

$$M = \frac{\tau}{(2\alpha+\beta)} + \xi .$$

From Section 3.3, the UMVUEs of $\xi$ and $\tau$ are $\left\{T_1 - \frac{T_2}{n(2n-1)(2\alpha+\beta)}\right\}$ and $\frac{T_2}{(2n-1)}$ respectively. Hence the UMVUE of M is

$$M^* = \left\{\frac{1}{(2\alpha+\beta)}\right\}\left(\frac{T_2}{2n-1}\right) + \left\{T_1 - \frac{T_2}{n(2n-1)(2\alpha+\beta)}\right\}.$$

$$= T_1 + \left\{\frac{(n-1)}{(2n-1)(2\alpha+\beta)}\right\} T_2.$$

**MREE** From Section 3.4, the MREE of $c\xi + d\tau$ is

$$M^{**} = c\,T_1 + \frac{1}{2n}\left\{d - \frac{c}{n(2\alpha+\beta)}\right\} T_2.$$

Here $c = 1$, $d = \frac{1}{(2\alpha+\beta)}$.

Then $M^{**} = T_1 + \left\{\frac{(n-1)}{2n^2(2\alpha+\beta)}\right\} T_2.$

**MLE** The MLE of $(\xi, \tau)$ is

$$\hat{\xi} = T_1 \text{ and } \hat{\tau} = \frac{1}{2n} T_2.$$

Then the MLE of MTBF is

$$\hat{M} = \frac{\hat{\tau}}{(2\alpha+\beta)} + \hat{\xi}.$$

The MLE of the reliability function is

$$\hat{R}(t) = \exp\left\{-\left(\frac{2\alpha+\beta}{\hat{\tau}}\right)(t-\hat{\xi})\right\},\ t > \hat{\xi}.$$

### 4.4.1 Standby system

**UMVUE** The MTBF is

$$M = \frac{4\alpha+3\beta}{(\alpha+\beta)(2\alpha+\beta)}\tau + 2\xi.$$

From Section 3.3, the UMVUE s of $\xi$ and $\tau$ are $\left\{T_1 - \frac{T_2}{n(2n-1)(2\alpha+\beta)}\right\}$ and $\frac{T_2}{(2n-1)}$ respectively. Hence the UMVUE of M is

$$M^* = \left\{\frac{4\alpha+3\beta}{(\alpha+\beta)(2\alpha+\beta)}\right\}\left(\frac{T_2}{2n-1}\right) + 2\left\{T_1 - \frac{T_2}{n(2n-1)(2\alpha+\beta)}\right\}$$

$$= 2T_1 + \left\{\frac{(4n-2)\alpha + (3n-2)\beta}{n(2n-1)(\alpha+\beta)(2\alpha+\beta)}\right\} T_2.$$

**MREE** From Section 3.4, the MREE of $c\xi + d\tau$ is

$$M^{**} = c\,T_1 + \frac{1}{2n}\left\{d - \frac{c}{n(2\alpha+\beta)}\right\}T_2.$$

Here $c = 2$, $d = \dfrac{4\alpha+3\beta}{(\alpha+\beta)(2\alpha+\beta)}$.

Then $M^{**} = 2T_1 + \left\{\dfrac{(4n-2)\alpha + (3n-2)\beta}{2n^2(\alpha+\beta)(2\alpha+\beta)}\right\}T_2.$

**MLE** The MLE of $(\xi, \tau)$ is

$$\hat{\xi} = T_1 \text{ and } \hat{\tau} = \frac{1}{2n}T_2.$$

Then the MLE of MTBF is

$$\hat{M} = \frac{4\alpha+3\beta}{(\alpha+\beta)(2\alpha+\beta)}\hat{\tau} + 2\hat{\xi}.$$

The MLE of the reliability function is

$$\hat{R}(t) = \left(\frac{2\alpha+2\beta}{\beta}\right)\exp\left\{-\left(\frac{2\alpha+\beta}{2\hat{\tau}}\right)(t-2\hat{\xi})\right\},$$

$$-\left(\frac{2\alpha+\beta}{\beta}\right)\exp\left\{-\left(\frac{\alpha+\beta}{\hat{\tau}}\right)(t-2\hat{\xi})\right\}t > 2\hat{\xi}.$$

# REFERENCES

Amala Revathy, S. (2003), Equivariant estimation of parameters based on sequential order statistics, *M.Phil Dissertation submitted to Loyola College (Autonomous), Chennai-600 034, University of Madras, India.*

Anuradha, A.C. (1991), Estimation of reliability for exponential models, *M.Phil Dissertation submitted to Loyola College (Autonomous), Chennai-600 034, University of Madras, India.*

Arnold, B.C. and Strauss, D.J. (1991), Bivariate distributions with conditionals in prescribed exponential families, *J.R. Stat. Soc. Ser. B Stat. Methodol.* **53**, 365 - 375.

Barlow, R.E. and Proschan, F. (1975), *Statistical theory of reliability and life testing: Probability models*, Holt, Rinehart and Winston Inc., New York.

Block, H.W. and Basu, A.P. (1974), A continuous bivariate exponential extension, *J. Amer. Statist. Assoc.*, **69**, 1031 - 1037.

Block, H.W. (1977), A characterization of a bivariate exponential distribution, *Ann. Statist.*, **5**, 808 - 812.

Chandrasekar, B (1984), A note on uniformly minimum risk unbiased estimator. *J. Ind. Statist. Assoc.*, **22**, 67 - 69.

Chandrasekar, B. and Amala Revathy, S. (2007), Equivariant estimation of parameters based on sequential order statistics for (1, 3) and (2, 3) systems, *Comm. Statist. Theory Methods*, **36**, 541 - 548.

Datta, G.S. and Ghosh, M. (1988), Minimum risk equivariant estimators of percentiles in location-scale families of distributions, *Calcutta Statist. Assoc. Bull.*, **37**, 147 - 148.

Edwin Prabakaran, T. and Chandrasekar, B. (1994), Simultaneous equivariant estimation for location-scale models, *J. Statist. Planning Infer.*, **40**, 51 - 59.

Freund, J. E. (1961), A bivariate extension of the exponential distribution, *J. Amer. Statist. Assoc.*, **56**, 971 - 977.

Gupta, R.L., Mehrotra, K.G. and Michalek, J.E. (1984), A small sample test for an absolutely continuous bivariate exponential model, *Comm. Statist. Theory Methods*, **A13**, 1735 - 1740.

Hanagal, D.D. and Kale, B.K. (1991), Large sample tests of independence in an absolutely continuous bivariate exponential distribution, *Comm. Statist. Theory Methods*, **20**, 1301 - 1313.

Hanagal, D.D. (1992), Some inference results in modified Freund's bivariate exponential distribution, *Biom. J.*, **34**, 745 -756.

Hanagal, D.D. (1993), Some inference results in an absolutely continuous multivariate exponential model of Block, *Statist. Probab. Letters*, **16**, 177 - 180.

Hanagal, D.D. (1995), Estimation of reliability in absolutely continuous bivariate exponential stress-strength models, *Econ. Qual. Control*, **10**, 217 - 222.

Hanagal, D.D. (1996), Estimation of system reliability from stress-strength relationship, *Comm. Statist. Theory Methods*, **25**, 1783 - 1797.

Hanagal, D.D. (1998), Estimation of system reliability in stress-strength models for distributions useful in life testing, *IAPQR Transactions*, **23**, 61 - 65.

Hanagal, D.D. (1999), Estimation of reliability of a component subjected to bivariate exponential stress, *Statist. Papers*, **40**, 211 - 220.

Kale, B.K. and Chandrasekar, B. (1983), On the equivalence of optimality criteria for vector unbiased statistics, *J. Ind. Statist. Assoc.*, **21**, 49 - 58.

Kotz, S. and Singpurwalla, N. D. (1999), On a bivariate distribution with exponential marginals, *Scand. J. Statist.*, **26**, 451.

Lehmann, E.L. (1986), *Testing Statistical Hypotheses*, Second Edn.; John Wiley and sons, New York.

Lehmann, E.L. and Casella, G (1998), *Theory of Point Estimation*, Second Edn.; Springer, New York.

Marshall, A.W. and Olkin, I. (1967a), A multivariate exponential distribution, *J. Amer. Statist. Assoc.*, **62**, 30 - 44.

Marshall, A.W and Olkin, I. (1967 b), A generalized bivariate exponential distribution, *J. Appl. Prob.*, **4**, 291 - 302.

Pitman, E.J.G (1937), The closest estimates of statistical parameters, *Math. Proc. Cambridge Philos. Soc.*, **33**, 212 - 222.

Richard Cowan (1987), A bivariate exponential distribution arising in random geometry, *Ann. Inst. Statist. Math.*, 103 - 111.

Ryu, K. (1993), An extension of Marshall and Olkin's bivariate exponential distribution, *J. Amer. Statist. Assoc.*, **87**, 1458 - 1465.

Sarkar, S.K. (1987), A continuous bivariate exponential distribution, *J. Amer. Statist. Assoc.*, **82**, 667 - 675.

Sukhatme, P.V. (1937), Tests of significance for samples of the $\chi^2$ population with two degrees of freedom, *Annals of Eugenics*, **8**, 52 - 56.

Usha, A. K. and Subramanyam, A. (2005), Tests of independence in a bivariate exponential distribution. *Metrika*, **61**.

Wang, C. and Ghosh, M. (2000), Bayesian analysis of bivariate competing risks models, *Sankhya* B. **62**, 388 - 401.

# Bivariate Distribution